The Squared Kemeny Rule for Averaging Rankings

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Introduction

Input: a profile of (strict) rankings $R_1, \dots, R_n$, possibly with weights $w_1, \dots, w_n$.
Output: a ranking R (or several rankings if there is a tie)

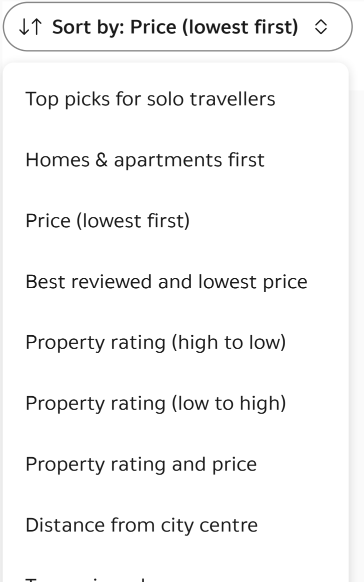
Well-known: Kemeny rule [1959] – choose R to minimize

$$\sum_{i \in N} w_i \cdot d(R, R_i)$$

where $d(R, R_i)$ is the Kendall-tau distance $|\{(a, b) : aRb \text{ and } bR_ia\}|$.

John G. Kemeny. "Mathematics without numbers". In: *Daedalus* 88.4 (1959), pp. 577–591. URL: https://www.jstor.org/stable/20026529

Application: ranking hotels



Application: ranking hotels

Price	90%	80%	70%	60%	50%	40%	30%	20%	10%	Score
La Quinta	?	?	?	?	?	?	?	?	?	Graduate
Graduate	?	?	?	?	?	?	?	?	?	The Study
NHH	?	?	?	?	?	?	?	?	?	H. Marcel
Omni	?	?	?	?	?	?	?	?	?	NHH
H. Marcel	?	?	?	?	?	?	?	?	?	Omni
The Study	?	?	?	?	?	?	?	?	?	La Quinta

Application: ranking hotels with Kemeny

Price	90%	80%	70%	60%	50%	40%	30%	20%	10%	Score
La Quinta	La Quinta	La Quinta	La Quinta	La Quinta	tied	Graduate	Graduate	Graduate	Graduate	Graduate
Graduate	Graduate	Graduate	Graduate	Graduate	tied	The Study	The Study	The Study	The Study	The Study
NHH	NHH	NHH	NHH	NHH	tied	H. Marcel	H. Marcel	H. Marcel	H. Marcel	H. Marcel
Omni	Omni	Omni	Omni	Omni	tied	NHH	NHH	NHH	NHH	NHH
H. Marcel	H. Marcel	H. Marcel	H. Marcel	H. Marcel	tied	Omni	Omni	Omni	Omni	Omni
The Study	The Study	The Study	The Study	The Study	tied	La Quinta	La Quinta	La Quinta	La Quinta	La Quinta

The Squared Kemeny Rule

Less well-known: Kemeny [1959] in the same article also introduced the "mean rule" which we call *Squared Kemeny* – choose R to minimize

$$\sum_{i \in N} w_i \cdot d(R, R_i)^2$$

where $d(R, R_i)$ is the Kendall-tau distance $|\{(a, b) : aRb \text{ and } bR_ia\}|$.

John G. Kemeny. "Mathematics without numbers". In: *Daedalus* 88.4 (1959), pp. 577–591. URL: https://www.jstor.org/stable/20026529

Application: ranking hotels with Squared Kemeny

Price	90%	80%	70%	60%	50%	40%	30%	20%	10%	Score
La Quinta	Graduate	Graduate	Graduate	Graduate	Graduate	Graduate	Graduate	Graduate	Graduate	Graduate
Graduate	La Quinta	La Quinta	La Quinta	NHH	NHH	NHH	H. Marcel	H. Marcel	H. Marcel	The Study
NHH	NHH	NHH	NHH	La Quinta	H. Marcel	H. Marcel	NHH	The Study	The Study	H. Marcel
Omni	Omni	H. Marcel	H. Marcel	H. Marcel	La Quinta	The Study	The Study	NHH	NHH	NHH
H. Marcel	H. Marcel	Omni	The Study	The Study	The Study	La Quinta	La Quinta	La Quinta	Omni	Omni
The Study	The Study	The Study	Omni	Omni	Omni	Omni	Omni	Omni	La Quinta	La Quinta

Other applications

- lists of products in e-commerce (ranking by cost, rating, delivery time, etc.)
- newsfeeds of social networks ("for you" versus "following")
- university rankings (student satisfaction, % of students employed after graduating, research output)
- voting: hiring committee needing to rank applicants (if you have too many ML colleagues, under Kemeny you might only ever hire more ML colleagues)
- groups of friends wanting to produce rankings of favorite music, restaurants, or travel destinations (majoritarian methods are weird in that context)

2-Rankings-Proportionality

A rule f satisfies *2-Rankings-Proportionality* (2RP) if for all profiles R

- consisting of just two rankings $\succ_1$ and $\succ_2$
- that disagree on $d = \text{swap}(\succ_1, \succ_2)$ pairwise comparisons,
- that have weights w_1 and w_2 with $w_1 + w_2 = 1$,

we have

$$f(R) = \{\rhd \in \mathcal{R} : \underbrace{d - \text{swap}(\succ_i, \rhd)}_{\substack{\text{number of agreements} \\ \text{between } \succ_i \text{ and the output}}} \in \text{round}(w_i \cdot d) \text{ for } i \in \{1, 2\}\},$$

where round(z) denotes the set of closest integers to z.

Example

Rankings $\succ_1$ and $\succ_2$ have weight 70% and 30% respectively, and they disagree on 10 pairwise comparisons. Then 2RP says the outcome should agree with $\succ_1$ on 7 of these comparisons, and with $\succ_2$ on the other comparisons.

Axiomatic Characterization

Theorem (Young–Levenglick, 1979)

The Kemeny rule is the only rule that satisfies

- *anonymity and neutrality*
- *reinforcement*
- *“Condorcet” property*

Theorem (this paper)

The Squared Kemeny rule is the only rule that satisfies

- *anonymity and neutrality*
- *reinforcement*
- *continuity*
- *2-ranking proportionality*

H. Peyton Young and Arthur Levenglick. “A consistent extension of Condorcet’s election principle”. In: *SIAM Journal on Applied Mathematics* 35.2 (1978), pp. 285–300. DOI: 10.1007/BFb0121198

Axiomatic Comparison

Kemeny rule

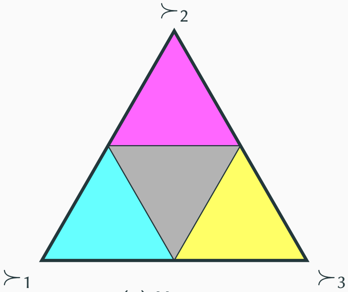
- ✓ Pareto efficiency
- ✓ participation
- ✓ strategyproofness (weak form)
- behaves like median on single-crossing profiles

Squared Kemeny rule

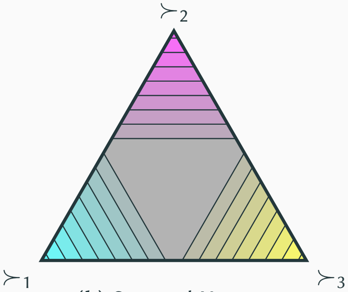
- ✓ Pareto efficiency
- ✓ participation
- ✗ strategyproofness
- behaves like mean on single-crossing profiles

30%	0%	30%	0%	10%	0%	0%	0%	0%	0%	30%
$\succ_0$	$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$	$\succ_5$	$\succ_6$	$\succ_7$	$\succ_8$	$\succ_9$	$\succ_{10}$
a	b	b	b	b	c	c	c	d	d	e
b	a	c	c	c	b	d	d	c	e	d
c	c	a	d	d	d	b	e	e	c	c
d	d	d	a	e	e	e	b	b	b	b
e	e	e	e	a	a	a	a	a	a	a

More than 2 rankings



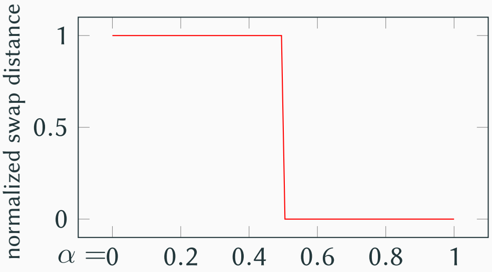
(a) Kemeny



(b) Squared Kemeny

Figure 1: The simplex of profiles in which the rankings $\succ_1 = abcdefgh$, $\succ_2 = fedcbahg$, and $\succ_3 = bahgfdec$ occur. Each point of the simplex is colored according to the swap distance of the (a) Kemeny and (b) Squared Kemeny ranking to the input rankings.

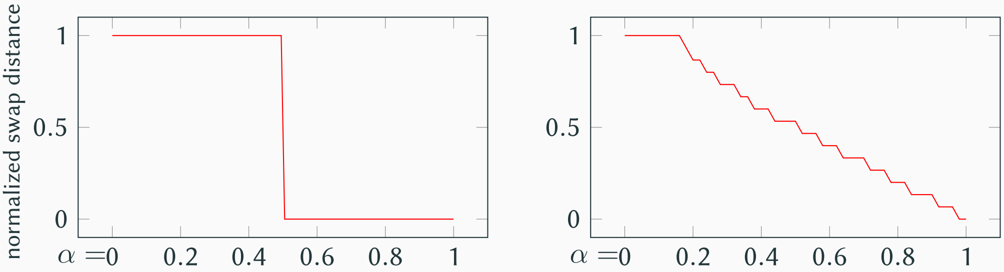
Comparing the Rules



(a) Kemeny

Figure 2: The maximum swap distance (normalized to [0, 1]) between the output of the Kemeny or Squared Kemeny rule to an input ranking, as a function of the weight α of the input ranking, for $m = 6$ alternatives.

Comparing the Rules



(a) Kemeny **(b)** Squared Kemeny

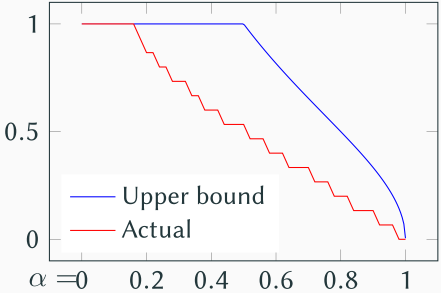
Figure 2: The maximum swap distance (normalized to $[0, 1]$) between the output of the Kemeny or Squared Kemeny rule to an input ranking, as a function of the weight α of the input ranking, for $m = 6$ alternatives.

Comparing the Rules

Theorem

If $\succ^* \in \mathcal{R}$ is an input ranking with weight α, and $\rhd$ is a Squared Kemeny output ranking, then

$$\mathsf{swap}(\succ^*, \rhd) \leqslant \sqrt{\frac{1-\alpha}{\alpha}} \cdot \binom{m}{2}.$$



In the paper: We also consider groups of voters who may have different rankings and prove that the SqK ranking cannot be too far away from them on average.

Computational Complexity

- Squared Kemeny is NP-complete to compute, even for $n = 4$ rankings. Similar reduction as for egalitarian Kemeny.

> Therese Biedl, Franz J. Brandenburg, and Xiaotie Deng. "On the complexity of crossings in permutations". In: *Discrete Mathematics* 309.7 (2009), pp. 1813–1823. DOI: 10.1016/j.disc.2007.12.088

- It can be computed using an ILP using the same trick as for maximizing Nash welfare in the allocation of indivisible goods.

> Ioannis Caragiannis et al. "The unreasonable fairness of maximum Nash welfare". In: *ACM Transactions on Economics and Computation (TEAC)* 7.3 (2019), pp. 1–32. DOI: 10.1002/cpa.3160130102

- The Kemeny ranking provides a 2-approximation to Squared Kemeny.
- For every constant $\varepsilon > 0$, there exists a polynomial-time $(2 + \varepsilon)$-approximation to the Squared Kemeny rule.

Computation

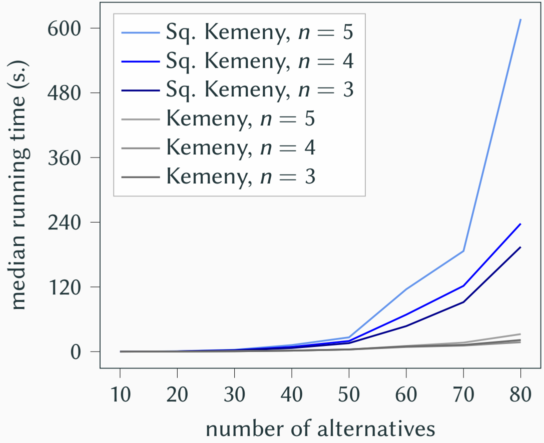


Figure 3: The median running time of computing the Squared Kemeny using Gurobi for a given number of alternatives and $n = \{3, 4, 5\}$ rankings occurring in the profile with equal weights, drawn uniformly at random. The values are based on 50 samples.

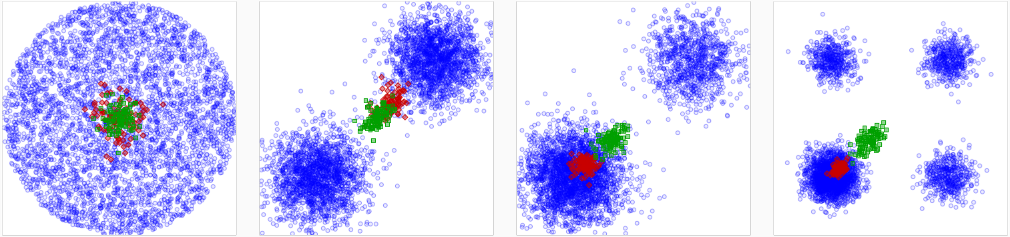


Figure 4: Euclidean embeddings. For the Kemeny rule, the positions of its outputs are denoted with a red diamond, and for Squared Kemeny with a green square.

Future Directions

- Draw "α-curves" for other social choice models.
- Figure out if there is a difference between "averaging" and "proportional aggregation", philosophically or formally.
- What about other distances such as the Spearman footrule distance?
- What about other p-Kemenys?
- What about using other types of rules? PAV? Multiple binary issues with a transitivity constraint?
- Can one make an epistemic argument for Squared Kemeny, e.g. as the MLE of a "normal" Mallows model?