Distribution Rules Under Dichotomous Preferences

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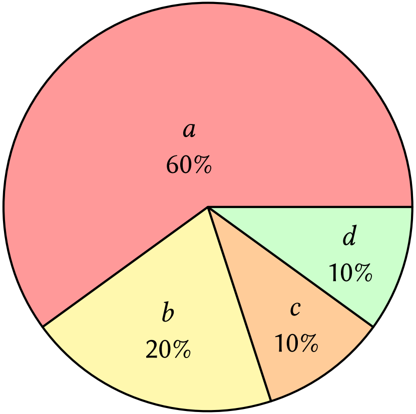
2023-10-10

Conference on Voting Theory and Preference Aggregation
Celebrating Klaus Nehring's 65th Birthday

ACM EC Conference 2021

Distribution rules

	a	b	c	d
Voter 1	✓			
Voter 2	✓		✓	
Voter 3	✓			✓
Voter 4		✓	✓	
Voter 5		✓		✓



Model

- Set of voters, $N = \{1, \ldots, n\}$.
- Set of projects $A = \{x_1, \ldots, x_m\}$.
- Possible outcomes $\Delta(A) = \{p : A \rightarrow [0, 1] : \sum_{x \in A} p_x = 1\}$.
- Each voter $i \in N$ approves project $A_i \subseteq A$.
- Voter gets utility $u_i(p) = \sum_{x \in A_i} p_x$ from distribution p.
- Voting rule takes the approval sets and outputs a distribution.

A. Bogomolnaia, H. Moulin, and R. Stong. "Collective choice under dichotomous preferences". In: *Journal of Economic Theory* 122.2 (2005), pp. 165–184

C. Duddy. "Fair sharing under dichotomous preferences". In: *Mathematical Social Sciences* 73 (2015), pp. 1–5

H. Aziz, A. Bogomolnaia, and H. Moulin. "Fair mixing: the case of dichotomous preferences". In: *Proceedings of the 20th ACM Conference on Economics and Computation (ACM-EC)*. 2019, pp. 753–781

A. Guerdjikova and K. Nehring. "Weighing Experts, Weighing Sources: The Diversity Value". Working paper. 2014

Applications

- *Randomization*
  - Interpretation of probability as lotteries.
  - Use randomization for fairness.
- *Repeated decisions*
  - Alternate projects for recurring decisions.
  - Example: Mix seminar days based on polls (10% Wed, 50% Thu, 40% Fri).
- *Budget division*
  - Decide budget division among projects via voting.
  - Non-monetary budgets: e.g., class time distribution based on student interests.
- *Approval-based apportionment*
- *Weighing criteria*
  - Organization has to make decisions in the future, based on multiple criteria. Voters say which criteria are important to them. (e.g. which students to admit)
- *Weighing experts*
  - Each competence or perspective is a (weighted) voter approving all experts with that competence. (e.g. Bundestagswahlrechtsreformausschuss)

Axioms

- *Efficiency:* When the rule selects p, there cannot be another distribution q with $u_i(q) \geqslant u_i(p)$ for all $i \in N$ and $u_i(q) > u_i(p)$ for some $i \in N$.
- *Strategyproofness*
- *Monotonicity:* If a voter starts approving x and nothing else changes, then p_x weakly increases.
- *Fairness axioms*
  - *Positive share*: $u_i(p) > 0$ for all $i \in N$.
  - *Individual fair share*: $u_i(p) \geqslant \frac{1}{n}$ for all $i \in N$.
  - *Group fair share*: For all $S \subseteq N$, writing $A_S = \bigcup_{i \in S} A_i$, we have $\sum_{x \in A_S} p_x \geqslant \frac{|S|}{|N|}$.
  - *Decomposability*: We can write $p = p_1 + \cdots + p_n$, where each p_i is a distribution summing to $\frac{1}{n}$ and only having support on i's approved projects.

Theorem

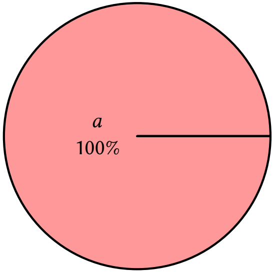
A distribution p is decomposable if and only if it satisfies group fair share.

Utilitarian rule

- Select a distribution p maximizing $\sum_{i \in N} u_i(p)$.
- Equivalent, put 100% on the approval winner(s).
- For concreteness, take uniform distribution on approval winners.

✓ **efficiency** is satisfied.
✗ **positive share** is failed.
✓ **strategyproofness** is satisfied, for the same reason that approval voting is strategyproof under dichotomous preferences.
✓ **monotonicity** is satisfied because strategyproofness implies monotonicity.
✓ **participation** is satisfied in weak versions.

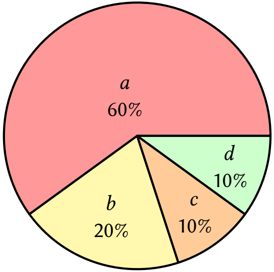
	a	b	c	d
Voter 1	✓			
Voter 2	✓		✓	
Voter 3	✓			✓
Voter 4		✓	✓	
Voter 5		✓		✓



Conditional utilitarian rule

- Select a distribution p maximizing $\sum_{i \in N} u_i(p)$ subject to p being decomposable.
- Equivalent, each agent $i \in N$ gets $1/n$ probability mass, and spreads it uniformly among projects that i approves and that have highest approval score.

	a	b	c	d
Voter 1	✓			
Voter 2	✓		✓	
Voter 3	✓			✓
Voter 4		✓	✓	
Voter 5		✓		✓



✗ **efficiency** is failed: in the example, $0.7a + 0.3b$ is a Pareto improvement. But no decomposable distribution can dominate!

✓ **decomposability** is satisfied.

✓ **strategyproofness** is satisfied.

✓ **monotonicity** is satisfied because strategyproofness implies monotonicity.

✓ **participation** is satisfied in strong versions.

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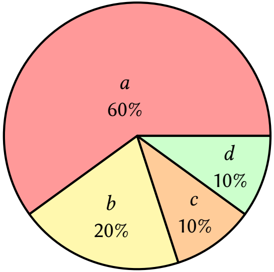
Voter 3	✓			✓
Voter 4		✓	✓	
Voter 5		✓		✓



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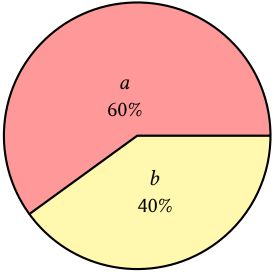
✓ **monotonicity** is satisfied because strategyproofness implies monotonicity.
✓ **participation** is satisfied in strong versions.

Nash rule

	a	b	c	d
Voter 1	✓			
Voter 2	✓		✓	
Voter 3	✓			✓
Voter 4		✓	✓	
Voter 5		✓		✓

- Select a distribution p maximizing $\prod_{i \in N} u_i(p)$.

✓ **efficiency** is satisfied.
✓ **decomposability** is satisfied.
✗ **strategyproofness** is failed.
✗ **monotonicity** is failed.
✓ **participation** is satisfied in strong versions.



Nash rule: axiomatic characterization

Nash rule is the unique rule that satisfies

- convex-valuedness, continuity
- reinforcement
- ex post dominance: if a project is dominated, it gets 0.
- exclusion: if we delete an alternative that gets 0, the result does not change.
- proportionality: be decomposable on profiles where every vote is a singleton

A. Guerdjikova and K. Nehring. "Weighing Experts, Weighing Sources: The Diversity Value". Working paper. 2014

Nash rule: decomposability and computation

- Nash satisfies decomposability, because it satisfies a cool fixed point property.
- Let p be the Nash outcome, and fix some $i \in N$. Let p_i be the distribution with

$$p_i(y) = \frac{1}{n} \cdot \frac{p_y}{\sum_{x \in A_i} p_x} \quad \text{for all } y \in A_i \text{, and 0 otherwise.}$$

- Then $p = p_1 + \cdots + p_n$.
- This suggests a "proportional response dynamic" for computing Nash (start with uniform distribution, then iterate). This converges (quite fast in practice).
- Nash is equivalent to Lindahl equilibrium from the theory of public goods.

A. Guerdjikova and K. Nehring. "Weighing Experts, Weighing Sources: The Diversity Value". Working paper. 2014

T. Cover. "An algorithm for maximizing expected log investment return". In: *IEEE Transactions on Information Theory* 30.2 (1984), pp. 369–373

B. Fain, A. Goel, and K. Munagala. "The core of the participatory budgeting problem". In: *Proceedings of the 12th International Conference on Web and Internet Economics (WINE)*. Lecture Notes in Computer Science (LNCS). Springer-Verlag, 2016, pp. 384–399

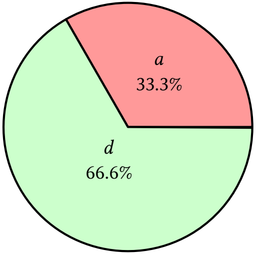
Nash rule: monotonicity

✗ **monotonicity** is failed.

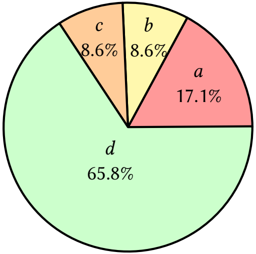
Smallest example has $m = 4$ and $n = 9$.

Have not found any examples with a “large” violation.

	a	b	c	d
Voter 1	✓			
Voter 2	✓	✓		
Voter 3	✓		✓	
Voter 4		✓	✓	✓
Voter 5		✓	✓	✓
Voter 6		✓		✓
Voter 7		✓		✓
Voter 8			✓	✓
Voter 9			✓	✓



	a	b	c	d
Voter 1	✓			✓
Voter 2	✓	✓		
Voter 3	✓		✓	
Voter 4		✓	✓	✓
Voter 5		✓	✓	✓
Voter 6		✓		✓
Voter 7		✓		✓
Voter 8			✓	✓
Voter 9			✓	✓



Axioms

	utilitarian	cond. utilitarian	Nash
efficiency	✓	–	✓
fairness	–	✓	✓
strategyproofness	✓	✓	–

Axioms

Theorem

No rule is anonymous, neutral, efficient, strategyproof, and satisfies individual fair share ($u_i(p) \geqslant \frac{1}{n}$) when $n \geqslant 5$ and $m \geqslant 17$.

> A. Bogomolnaia, H. Moulin, and R. Stong. "Collective choice under dichotomous preferences". In: *Journal of Economic Theory* 122.2 (2005), pp. 165–184

Quotes: "We submit as a challenging conjecture the following statement: there is no strategyproof and *ex ante* efficient mechanism guaranteeing positive shares", "we suspect the answer is negative when [the numbers of agents and projects] are large enough", "we have not been able to determine if one of the anonymity or neutrality property (or both) can be dropped."

Surprisingly simple

Theorem

No rule is anonymous, neutral, efficient, strategyproof, and satisfies positive share ($u_i(p) > 0$) when $n \geqslant 5$ and $m \geqslant 4$.

	a	b	c	d
Voter 1	✓			
Voter 2	✓		✓	
Voter 3	✓			✓
Voter 4		✓	✓	
Voter 5	✓	✓		

b and c are symmetric, so get same share.
We must have $p_b = p_c > 0$ by positive share for Voter 4.
Hence we have $u_5(p) < 1$.
Now suppose voter 5 approves d instead of a.

	a	b	c	d
Voter 1	✓			
Voter 2	✓		✓	
Voter 3	✓			✓
Voter 4		✓	✓	
Voter 5		✓		✓

c and d are symmetric, so get same share.
If $p_c = p_d = \epsilon > 0$, we can move ϵ from c to a and ϵ from d to b to get a Pareto improvement.
So $p_c = p_d = 0$, and thus $p_a + p_b = 1$.
Hence voter 5 manipulated successfully.

Automatically getting an impossibility

- Could make an LP: Generate all profiles with 5 voters and 4 alternatives, add variables encoding the distribution selected by voting rule.
- Constraints for strategyproofness and positive share: easy. But how to do efficiency?
- Theorem: Whether a distribution is efficient depends only on its support, and efficient supports can be found in poly time.
- So one can use binary variables to encode efficiency.
- But it doesn't scale very well. A discrete encoding would be better.

SAT solving

- Note: efficiency and positive share only depend on support $\rightarrow$ discrete problem.
- But what about strategyproofness?
- Idea: Weaken strategyproofness ($\rightarrow$ stronger impossibility)
- Use *pessimistic* strategyproofness: Manipulation is only successful if we go from utility 0 to > 0 of from < 1 to 1.
- This depends only on support.
- Now we can use SAT solving.

Theorem

No rule is efficient, strategyproof, and satisfies positive share ($u_i(p) > 0$) when $n \geqslant 6$ and $m \geqslant 4$.

Proof goes through 386 profiles.

	A_1	A_2	A_3	A_4	A_5	A_6	possible supports	dominated supports
Profile 1	b	c	ab	ac	bd	cd	$\underline{bc}, \underline{abc}, bcd$	$ad \leftarrowtail bc$
Profile 2	b	c	abc	ac	bd	cd	$\underline{bc}, bcd$	$a \leftarrowtail c,\ ab \leftarrowtail bc,\ ad \leftarrowtail bc$
Profile 3	b	c	bc	ac	bd	cd	$\underline{bc}, bcd$	$a \leftarrowtail c,\ ab \leftarrowtail bc,\ ad \leftarrowtail bc$
Profile 4	bc	c	bc	ac	bd	cd	$cd, \underline{bc}, bcd$	$a \leftarrowtail c,\ ab \leftarrowtail bc,\ ad \leftarrowtail bc$
Profile 5	bc	c	bc	ac	bd	acd	$cd, \underline{bc}, \underline{bcd}$	$a \leftarrowtail c,\ ab \leftarrowtail bc,\ ad \leftarrowtail cd$
Profile 6	bc	c	bc	ac	bd	ad	$cd, acd, \underline{bcd}$	$ab \leftarrowtail cd$
Profile 7	bc	c	bc	ac	bcd	ad	$ac, \underline{cd}, acd$	$b \leftarrowtail c,\ ab \leftarrowtail ac,\ bd \leftarrowtail cd$
Profile 8	bc	c	bc	ac	cd	ad	$ac, \underline{cd}, acd$	$b \leftarrowtail c,\ ab \leftarrowtail ac,\ bd \leftarrowtail ac$
…								
Profile 190	b	bc	ab	abc	bd	cd	$bc, \underline{bd}, \underline{bcd}$	$a \leftarrowtail b,\ ac \leftarrowtail bc,\ ad \leftarrowtail bc$
Profile 191	b	c	ab	abc	bd	cd	$bc, \underline{bd}, \underline{bcd}$	$a \leftarrowtail b,\ ac \leftarrowtail bc,\ ad \leftarrowtail bc$
Profile 1	b	c	ab	ac	bd	cd	$bc, abc, \underline{bcd}$	$ad \leftarrowtail bc$

Axioms

	utilitarian	cond. utilitarian	Nash
efficiency	✓	–	✓
fairness	–	✓	✓
monotonicity	✓	✓	–

Another impossibility?

Designing efficient rules

- Reinforcement characterization "implies" that Nash is the only decomposable rule that maximizes a separable function of voter utility.

> A. Guerdjikova and K. Nehring. "Weighing Experts, Weighing Sources: The Diversity Value". Working paper. 2014

- Among rules of the form "choose p that maximizes $\sum_{i\in N} g(u_i(p))$", only $g = \log$ (i.e., Nash) satisfies group fair share. (And only $g = \mathrm{id}$ satisfies strategyproofness.)

> A. Bogomolnaia, H. Moulin, and R. Stong. "Collective choice under dichotomous preferences". Working paper. 2002

- But how else to design an efficient rule?
- Theorem: A distribution p is Pareto efficient if and only if there are positive weights $(w_i)_{i\in N}$ such that p maximizes $\sum_{i\in N} w_i \cdot u_i(p)$.
- Idea: Given a profile, vary weights until we get a decomposable distribution. Hopefully vary the weights in a way that gives a monotonic rule.

Sequential utilitarian rule

- Note that p maximizes $\sum_{i \in N} w_i \cdot u_i(p)$ iff its support consists only of projects with maximum weighted approval score.
- Start with $w_i = 1$ for all $i \in N$.
- Repeatedly:
  - For every voter who approves a w-maximum projects, we assign $\frac{1}{n}$ to those projects, and freeze these contributions.
  - Then we continuously increase the weights of all unassigned voters until a new project becomes w-maximum.

Theorem

The sequential utilitarian rule is monotonic.

However it fails participation. Smallest known example has $m = 5$ and $n = 45$. No counterexamples for $m = 4$ and $n \leqslant 14$, or for $m = 5$ and $n \leqslant 10$.

Other relaxations of strategyproofness

- Subset strategyproofness. Agents are only allowed to manipulate by reporting a subset of their true approval set.
- Impossibility still holds (with anonymity and neutrality, in 1 step)
- Superset strategyproofness. Agents are only allowed to manipulate by reporting a superset of their true approval set.
- Nash and sequential utilitarian fail this. Unknown if there is an efficient and decomposable rule satisfying this
- But leximin does satisfy it Leximin even satisfies excludable strategyproofness.

H. Aziz, A. Bogomolnaia, and H. Moulin. "Fair mixing: the case of dichotomous preferences". In: *Proceedings of the 20th ACM Conference on Economics and Computation (ACM-EC)*. 2019, pp. 753–781

X. Bei, X. Lu, and W. Suksompong. "Truthful cake sharing". In: *Proceedings of the 36th AAAI Conference on Artificial Intelligence (AAAI)*. 2022, pp. 4809–4817

Axioms

	utill.	leximin	cond. util.	Nash	seq. util.	No Rule!
Efficiency	✓	✓	–	✓	✓	↯
↳ Decomposable Efficiency	✓	✓	✓	✓	✓	
Decomposability (GFS)	–	–	✓	✓	✓	
↳ Positive Share	–	✓	✓	✓	✓	↯
Strategyproofness	✓	–	✓	–	–	↯
↳ Monotonicity	✓	–	✓	–	✓	
Contribution IC	–	–	✓	✓	–	
↳ Weak Participation	✓	✓	✓	✓	–	

Other points

- Cake sharing.
- Welfare loss due to fairness: Nash and CUT obtain at least a $\frac{2}{\sqrt{m}}$ fraction of optimum utilitarian welfare.
- Linear utilities, rankings.

M. Michorzewski, D. Peters, and P. Skowron. "Price of Fairness in Budget Division and Probabilistic Social Choice". In: *Proceedings of the 34th AAAI Conference on Artificial Intelligence (AAAI)*. Forthcoming. 2020

S. Airiau, H. Aziz, I. Caragiannis, J. Kruger, J. Lang, and D. Peters. "Portioning using Ordinal Preferences: Fairness and Efficiency". In: *Artificial Intelligence* 314 (2023), p. 103809

S. Ebadian, A. Kahng, D. Peters, and N. Shah. "Optimized distortion and proportional fairness in voting". In: *Proceedings of the 23rd ACM Conference on Economics and Computation (EC)*. 2022, pp. 563–600

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