Computing Lindahl Equilibrium for Public Goods with and without Funding Caps

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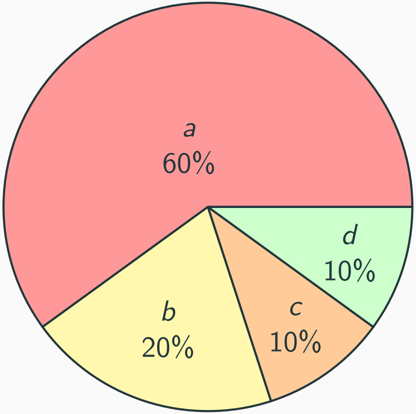
Joint work with Christian Kroer (Columbia)

Distribution rules

	a	b	c	d
Voter 1	✓			
Voter 2	✓		✓	
Voter 3	✓			✓
Voter 4		✓	✓	
Voter 5		✓		✓



Model of Fair Mixing / Portioning / Uncapped Public Goods

- Set of public goods or projects M, and an available budget $B > 0$.
- Set of voters $N = \{1, \dots, n\}$ each with an endowment B_i where $\sum_{i \in N} B_i = B$.
    - often: $B_i = B/n$, equal endowments.
- Voter $i \in N$ has valuation $v_{ij} \geqslant 0$ for good j (special case: approval with $v_{ij} \in \{0, 1\}$)
- An allocation is a vector $x = (x_j)_{j \in M}$ with $x_j \geq 0$ for all $j \in M$ and $\sum_{j \in M} x_j \leqslant B$.
- Voter gets utility $u_i(x) = \langle v_i, x \rangle = \sum_{j \in M} v_{ij} x_j$ from outcome x.

A. Bogomolnaia, H. Moulin, and R. Stong. "Collective choice under dichotomous preferences". In: *Journal of Economic Theory* 122.2 (2005), pp. 165–184

C. Duddy. "Fair sharing under dichotomous preferences". In: *Mathematical Social Sciences* 73 (2015), pp. 1–5

H. Aziz, A. Bogomolnaia, and H. Moulin. "Fair mixing: the case of dichotomous preferences". In: *Proceedings of the 20th ACM Conference on Economics and Computation (ACM-EC)*. 2019, pp. 753–781

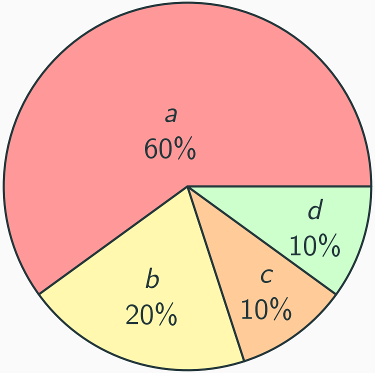
Ani Guerdjikova and Klaus Nehring. "Weighing experts, weighing sources: The diversity value". In: (2014). Working Paper. URL: https://dominik-peters.de/archive/guerdjikova2014.pdf

Applications

- *Randomization*
  - Interpretation of probability as lotteries.
  - Use randomization for fairness.
- *Repeated decisions*
  - Alternate projects for recurring decisions.
  - Example: Mix seminar days based on polls (10% Wed, 50% Thu, 40% Fri), mix restaurants to go lunch to.
- *Budget division*
  - Decide budget division among projects via voting.
  - Non-monetary budgets are also possible: e.g., class time distribution based on student interests.
- *Approval-based apportionment*
- *Weighing criteria*
  - Organization has to make decisions in the future, based on multiple criteria. Voters say which criteria are important to them.

Fairness

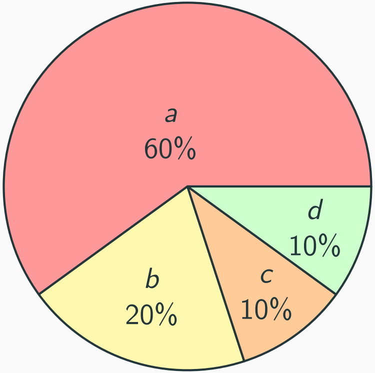
	a	b	c	d
Voter 1	✓			
Voter 2	✓		✓	
Voter 3	✓			✓
Voter 4		✓	✓	
Voter 5		✓		✓



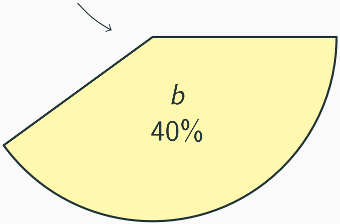
Is this a fair outcome?

Fairness

	a	b	c	d
Voter 1	✓			
Voter 2	✓		✓	
Voter 3	✓			✓
Voter 4		✓	✓	
Voter 5		✓		✓



Voters 4 and 5 are 40% of population and could propose this distribution with "their" share of the budget and they both prefer this one



Is this a fair outcome?

The Core

An allocation x is blocked by a coalition $S \subseteq N$ if it can propose an objection $z = (z_j)_{j \in M}$ with $\sum_{j \in M} z_j \leqslant \sum_{i \in S} B_i$ such that $u_i(z) \geqslant u_i(x)$ for all $i \in S$, with at least one inequality strict.

An allocation x is in the core if it is not blocked by any coalition.

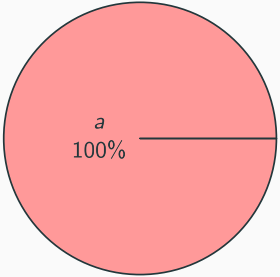
Note: If x is in the core then

- it is Pareto optimal (consider $S = N$) and
- it satisfies individual fair share: $u_i(x) \geqslant B_i \cdot \max_{j \in M} v_{ij}$ for all $i \in N$ (consider $S = \{i\}$).

Different rules

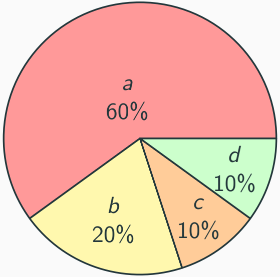
	a	b	c	d
Voter 1	✓			
Voter 2	✓		✓	
Voter 3	✓			✓
Voter 4		✓	✓	
Voter 5		✓		✓

Utilitarian



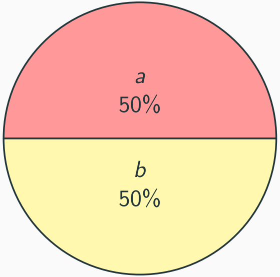
blocked by $S = \{4, 5\}$

Conditional Utilitarian (CUT)



blocked by $S = \{4, 5\}$

Egalitarian



blocked by $S = \{1, 2, 3\}$

Nash rule

The Nash rule maximizes Nash social welfare:

- Select an allocation x maximizing $\prod_{i \in N} u_i(x)^{B_i}$.
- equivalently: maximizing $\sum_{i \in N} B_i \log u_i(x)$.

Theorem

The Nash rule selects an allocation in the core.

On the next slide, we will see why Nash satisfies the core.

Brandon Fain, Ashish Goel, and Kamesh Munagala. “The core of the participatory budgeting problem”. In: *Proceedings of the 12th International Conference on Web and Internet Economics (WINE)*. 2016, pp. 384–399. DOI: 10.1007/978-3-662-54110-4_27

H. Aziz, A. Bogomolnaia, and H. Moulin. “Fair mixing: the case of dichotomous preferences”. In: *Proceedings of the 20th ACM Conference on Economics and Computation (ACM-EC)*. 2019, pp. 753–781

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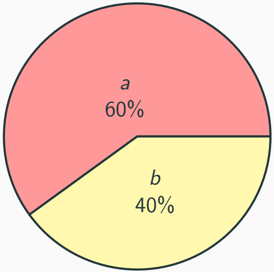


Lindahl Equilibrium

Let x be an allocation and let $p = (p_{ij})_{i \in N, j \in M} \geqslant 0$ be a collection of personalized prices.

Then (x, p) is a *Lindahl equilibrium* if

- x is affordable: we have $\langle p_i, x \rangle \leqslant B_i$ for every $i \in N$,
- x is utility-maximizing: for every $i \in N$ and all $y \in \mathbb{R}^m_{\geqslant 0}$ with $\langle p_i, y \rangle \leqslant B_i$, $u_i(x) \geqslant u_i(y)$,
- x is profit-maximizing: for every $j \in M$, $\sum_{i \in N} p_{ij} \leqslant 1$, and if $x_j > 0$ then $\sum_{i \in N} p_{ij} = 1$.

Every agent demands the same public bundle.

We can view $p_i j$ as the fraction of x_j that i pays for, so the spending of agent i towards good j is $b_{ij} := p_{ij} x_j$.

Lindahl Equilibrium: Theorems

Theorem

A Lindahl equilibrium exists for strictly monotone convex utilities.

Proof via fixed point theorem (reduction to Arrow-Debreu).

Theorem

If (x, p) is a Lindahl equilibrium then x is in the core.

Duncan K. Foley. "Lindahl's solution and the core of an economy with public goods". In: *Econometrica* 38.1 (1970), pp. 66–72. DOI: 10.2307/1909241

Theorem

An allocation x maximizes Nash welfare if and only if there exist prices p such that (x, p) is a Lindahl equilibrium.

Brandon Fain, Ashish Goel, and Kamesh Munagala. "The core of the participatory budgeting problem". In: *Proceedings of the 12th International Conference on Web and Internet Economics (WINE)*. 2016, pp. 384–399. DOI: 10.1007/978-3-662-54110-4_27

Lindahl Equilibrium: Example

Let's consider a simple example where $N = M$ and every agent likes only their own project:

	B_i	Project 1	Project 2	$\cdots$	Project n
Agent 1	B_1	1	0	$\cdots$	0
Agent 2	B_2	0	1	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
Agent n	B_n	0	0	$\cdots$	1

In a Lindahl equilibrium, by utility maximization, each agent spends all their money on their own project, so $x_i = B_i / p_{ii}$ and $x_i > 0$ for each i. Thus $p_{ij} = 0$ whenever $i \neq j$ since otherwise agent i would not demand $x_j > 0$.

By profit maximization, since $x_i > 0$, we have $p_{ii} = 1$.

Hence in Lindahl equilibrium, $x_i = B_i$ – the proportional outcome.

Nash rule: Computation

How to compute a Nash allocation?

Annoyingly, there are examples with a unique Nash allocation that is irrational:

	B_i	Project 1	Project 2	Project 3
Agent 1	0.25	1	0	0
Agent 2	0.25	1	0	1
Agent 3	0.25	1	1	0
Agent 4	0.25	0	1	1

In the Nash optimal allocation, $x_2 = x_3 = \frac{1}{16}(7 - \sqrt{17}) \approx 0.1798$.

Thus, we can only hope to compute an approximate Nash allocation.

Nash rule: Computation

How to compute a Nash allocation?

Most straightforward way: solve a convex program using some convex programming solver.

$$\max_{x \geq 0} \sum_{i \in N} B_i \log \langle v_i, x \rangle$$
$$\text{s.t.} \sum_{j \in M} x_j \leq B$$

But it would be nicer to have a simple algorithm and one with runtime guarantees.

Nash rule: Computation via dynamics

- There exists a "proportional response dynamics" that converges to the Nash distribution.
- At each iteration t, the dynamics has some current allocation $x^t = (x_1^t, \dots, x_j^t)$. Then the next budget allocation in the dynamics is

$$x_j^{t+1} = \sum_{i \in N} B_i \frac{v_{ij} x_j^t}{\langle v_i, x^t \rangle} \quad \text{for all } j \in M.$$

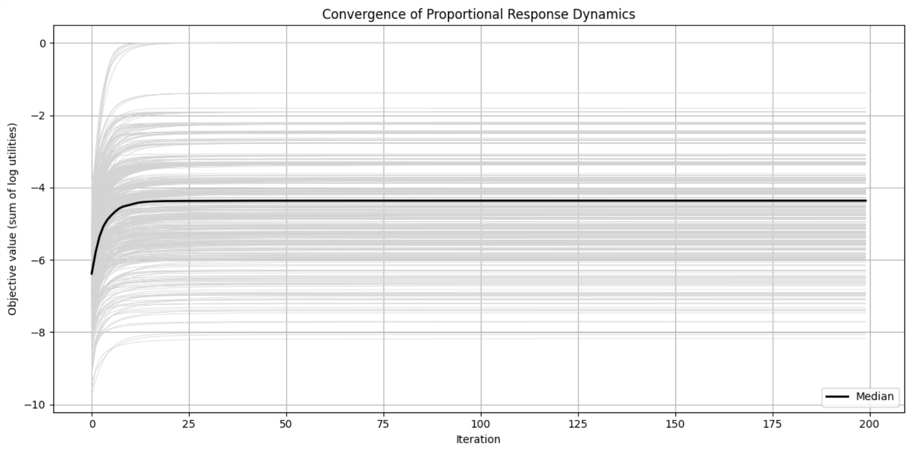
- This dynamics can be interpreted as each agent i independently deciding how they wish to allocate their share of the budget B_i in the next round. Specifically, agent i allocates spending proportional to how much utility each project provided them at round t.
- From KKT conditions, we know that Nash forms a fixed point of the dynamics.
- The dynamics converges to the optimum – a fact that was rediscovered multiple times.

Thomas Cover. "An algorithm for maximizing expected log investment return". In: *IEEE Transactions on Information Theory* 30.2 (1984), pp. 369–373. DOI: 10.1109/TIT.1984.1056869

Florian Brandl, Felix Brandt, Matthias Greger, Dominik Peters, Christian Stricker, and Warut Suksompong. "Funding public projects: A case for the Nash product rule". In: *Journal of Mathematical Economics* 99 (2022), p. 102585. DOI: 10.1016/j.jmateco.2021.102585

Nash rule: Convergence

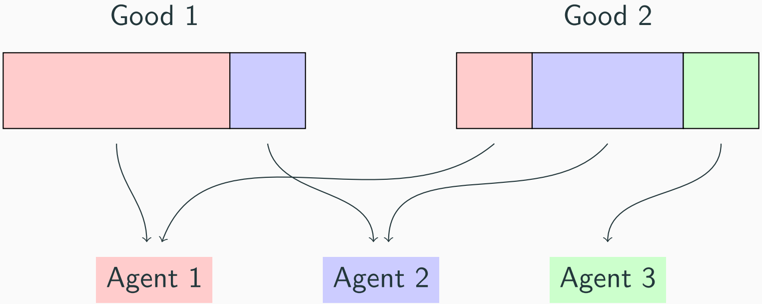
- Empirically, the dynamics converges very fast.



- But for 40 years, no bound on the convergence rate was known!
- To get a rate, let's get help from another area of social choice...

Allocation of homogeneous divisible goods

- Consider the model of private goods, where now an allocation $x = (x_{ij})$ specifies how much of each good each agent receives, such that $\sum_{i \in N} x_{ij} = 1$ for all $j \in M$.



Allocation of homogeneous divisible goods

- Consider the model of private goods, where now an allocation $x = (x_{ij})$ specifies how much of each good each agent receives, such that $\sum_{i \in N} x_{ij} = 1$ for all $j \in M$.
- In this model, the Nash rule is much better understood – an eerily similar!
- Nash is also fair (in the sense of envy-freeness instead of core).
- Nash is also a market equilibrium (Fisher market instead of Lindahl).

> Hal R Varian. "Equity, envy, and efficiency". In: *Journal of Economic Theory* 9.1 (1974), pp. 63–91

- "Fisher: same prices, individual demands; Lindahl: individual prices, same demands"
- Nash is also computable using a convex program.

> Edmund Eisenberg and David Gale. "Consensus of subjective probabilities: The pari-mutuel method". In: *The Annals of Mathematical Statistics* 30.1 (1959), pp. 165–168. DOI: 10.1214/aoms/1177706369

- Nash is also computable using a proportional response dynamics.

> Fang Wu and Li Zhang. "Proportional response dynamics leads to market equilibrium". In: *Proceedings of the 39th Annual ACM Symposium on Theory of Computing (STOC)*. 2007, pp. 354–363. DOI: 10.1145/1250790.1250844

- The proportional response dynamics has a known convergence rate!

Proportional Response Dynamics and the Shmyrev program

V. I. Shmyrev found a convex program that also computes a Fisher market equilibrium and that looks quite different from the classic Eisenberg–Gale program for maximizing Nash welfare.

The Eisenberg–Gale program.

$$\max_{x \geq 0} \sum_{i \in N} B_i \log \langle v_i, x_i \rangle$$

$$\text{s.t.} \sum_{i \in N} x_{ij} \leq 1 \quad \text{for all } j \in M$$

The Shmyrev program.

$$\max_{b \geq 0, p \geq 0} \sum_{i \in N, j \in M} b_{ij} \log v_{ij} - \sum_{j \in M} p_j \log p_j$$

$$\text{s.t.} \sum_{j \in M} b_{ij} = B_i \quad \text{for all } i \in N$$

$$\sum_{i \in N} b_{ij} = p_j \quad \text{for all } j \in M$$

The final allocation is $x_{ij} = b_{ij}/p_j$.

Vadim I. Shmyrev. "On an approach to the determination of equilibrium in elementary exchange models". In: *Doklady Akademii Nauk SSSR*. vol. 268:5. (In Russian). 1983, pp. 1062–1066. URL: https://www.mathnet.ru/eng/dan10141

Vadim I. Shmyrev. "An algorithm for finding equilibrium in the linear exchange model with fixed budgets". In: *Journal of Applied and Industrial Mathematics* 3.4 (2009), p. 505. DOI: 10.1134/S1990478909040097

Proportional Response Dynamics and the Shmyrev program

Birnbaum et al. (2011) showed that applying the standard mirror descent algorithm to the Shmyrev program is the same algorithm as computing the proportional response dynamics, and were thereby able to deduce a $1/t$ convergence rate bound.

> Benjamin Birnbaum, Nikhil R. Devanur, and Lin Xiao. "Distributed algorithms via gradient descent for Fisher markets". In: *Proceedings of the 12th ACM Conference on Electronic Commerce (EC)*. 2011, pp. 127–136. DOI: 10.1145/1993574.1993594

It was also shown that the Shmyrev program can be obtained by taking the dual of the Eisenberg–Gale program, rewriting it, and taking another dual.

> Richard Cole, Nikhil R. Devanur, Vasilis Gkatzelis, Kamal Jain, Tung Mai, Vijay V. Vazirani, and Sadra Yazdanbod. "Convex program duality, Fisher markets, and Nash social welfare". In: *Proceedings of the 2017 ACM Conference on Economics and Computation (EC)*. 2017, pp. 459–460. DOI: 10.1145/3033274.3085109

Back to public goods

Can we find analogs of these results for our public goods setting? It turns out we can!

The Nash welfare program.

$$\max_{x \geq 0} \sum_{i \in N} B_i \log \langle v_i, x \rangle$$
$$\text{s.t.} \sum_{j \in M} x_j \leq B$$

Our new program.

$$\max_{b \geqslant 0, x \geqslant 0} \sum_{i \in N, j \in M_i} b_{ij} \log v_{ij} - \sum_{i \in N, j \in M_i} b_{ij} \log \frac{b_{ij}}{x_j}$$
$$\text{s.t.} \sum_{j \in M_i} b_{ij} = B_i \quad \text{for all } i \in N$$
$$\sum_{i \in N_j} b_{ij} = x_j \quad \text{for all } j \in M$$

- Our new program also computes a Lindahl equilibrium.
- It can be obtained from the Nash program via double duality.
- The proportional response dynamics is equivalent to mirror descent on our new program, yielding a $1/t$ convergence rate.

Other divisible public goods models

- The “fair mixing” model we have looked at thus far is only one model of making public decisions in a divisible setting.
- Several related models have recently been studied:
- Fractional committee selection.

Mashbat Suzuki and Jeremy Vollen. “Maximum flow is fair: A network flow approach to committee voting”. In: *Proceedings of the 25th ACM Conference on Economics and Computation (EC)*. 2024, pp. 964–983. DOI: 10.1145/3670865.3673603

- Cake sharing.

Xiaohui Bei, Xinhang Lu, and Warut Suksompong. “Truthful cake sharing”. In: *Social Choice and Welfare* 64 (2025), pp. 309–343. DOI: 10.1007/s00355-023-01503-0

- Fractional participatory budgeting.

Kamesh Munagala, Yiheng Shen, Kangning Wang, and Zhiyi Wang. “Approximate core for committee selection via multilinear extension and market clearing”. In: *Proceedings of the 2022 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*. 2022, pp. 2229–2252. DOI: 10.1137/1.9781611977073.89

Fractional committee selection

In recent years, a very large literature on approval-based committee elections (ABC) has developed, where voters approve some candidates and the task is to elect k out of m candidates. Lots of representation axioms (JR, PJR, EJR, core, etc.) and voting rules (PAV/Thiele, Phragmén, Method of Equal Shares, etc.) have been proposed.

It is natural to consider a randomized version of the model, with the hope of obtaining "best-of-both-worlds" results.

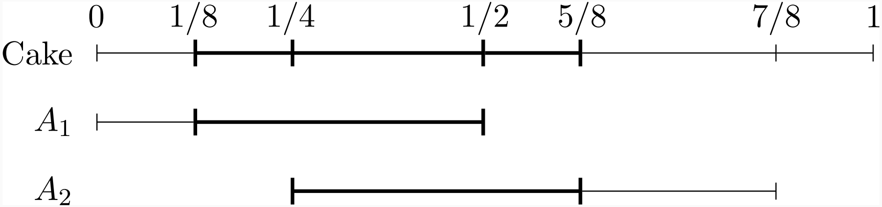
A fractional committee is a map $W : C \to [0, 1]$ with $\sum_{c \in C} W(c) = k$.

Haris Aziz, Xinhang Lu, Mashbat Suzuki, Jeremy Vollen, and Toby Walsh. "Best-of-both-worlds fairness in committee voting". In: *Proceedings of the 19th Conference on Web and Internet Economics (WINE)*. 2023. URL: https://arxiv.org/abs/2303.03642

Mashbat Suzuki and Jeremy Vollen. "Maximum flow is fair: A network flow approach to committee voting". In: *Proceedings of the 25th ACM Conference on Economics and Computation (EC)*. 2024, pp. 964–983. DOI: 10.1145/3670865.3673603

Cake sharing

In the cake sharing model, we can instantiate some fraction B of the cake $[0,1]$, for example $B = \frac{1}{2}$. Agents approve certain parts of the cake but not others.



Xiaohui Bei, Xinhang Lu, and Warut Suksompong. "Truthful cake sharing". In: *Social Choice and Welfare* 64 (2025), pp. 309–343. DOI: 10.1007/s00355-023-01503-0

Capped public goods setting

- All three models (fractional committees, cake sharing, fractional participatory budgeting) can be unified in a single model, which we call the capped public goods setting.
- For each public good $j \in M$, introduce an upper bound $\text{cap}_j \geqslant 0$.
- An allocation is now a vector $x = (x_j)_{j \in M}$ with $0 \leqslant x_j \leqslant \text{cap}_j$ for all $j \in M$ and $\sum_{j \in M} x_j \leqslant B$.
- The core and Lindahl equilibrium can be defined analogously by incorporating the cap constraints.
- A Lindahl equilibrium is known to exist via fixed point theorems (with some caveats since they require strict monotonicity).

Duncan K. Foley. "Lindahl's solution and the core of an economy with public goods". In: *Econometrica* 38.1 (1970), pp. 66–72. DOI: 10.2307/1909241

Capped public goods setting: The core

- A Lindahl equilibrium is known to exist via fixed point theorems (with some caveats since they require strict monotonicity). Thus, a core outcome exists.

Duncan K. Foley. "Lindahl's solution and the core of an economy with public goods". In: *Econometrica* 38.1 (1970), pp. 66–72. DOI: 10.2307/1909241

- But Nash is not equivalent to the core anymore![1]

	B_i	Project 1	Project 2	Project 3	Project 4
Agent 1	2	1	1	0	0
Agent 2	2	1	0	1	0
Agent 3	2	0	0	0	1
cap_j		3	∞	∞	∞

Nash allocation: $x = (3, 0, 0, 3)$. This allocation violates the core: consider the blocking coalition $S = \{1, 2\}$ and the objection $z = (3, 0.5, 0.5, 0)$.

[1] This example is similar to a well-known instance in ABC voting where PAV fails the core.

Capped public goods setting: The core

- Computing a Lindahl equilibrium and a core outcome was an open problem. "Is computing the Lindahl equilibrium for public goods computationally hard or is there a polynomial time algorithm even [when the public goods are capped]?"

> Brandon Fain, Ashish Goel, and Kamesh Munagala. "The core of the participatory budgeting problem". In: *Proceedings of the 12th International Conference on Web and Internet Economics (WINE)*. 2016, pp. 384–399. DOI: 10.1007/978-3-662-54110-4_27

- "there is no known polynomial time algorithm for computing fractional core"

> Mashbat Suzuki and Jeremy Vollen. "Maximum flow is fair: A network flow approach to committee voting". In: *Proceedings of the 25th ACM Conference on Economics and Computation (EC)*. 2024, pp. 964–983. DOI: 10.1145/3670865.3673603

- Munagala et al. (2022) studied *indivisible* public goods and approximate core. Rounding a Lindahl equilibrium "yields a 9.3-core, though we do not know how to implement the resulting algorithm in polynomial time". Without Lindahl: only 67-approximation.

> Kamesh Munagala, Yiheng Shen, Kangning Wang, and Zhiyi Wang. "Approximate core for committee selection via multilinear extension and market clearing". In: *Proceedings of the 2022 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*. 2022, pp. 2229–2252. DOI: 10.1137/1.9781611977073.89

Capped public goods setting: Our new program

- While Nash fails the core when adding cap constraints, it turns out that our new program continues to work!

$$
\begin{aligned}
\max_{b \geqslant 0, x \geqslant 0} \quad & \sum_{i \in N, j \in M_i} b_{ij} \log v_{ij} - \sum_{i \in N, j \in M_i} b_{ij} \log \frac{b_{ij}}{x_j} \\
\text{s.t.} \quad & \sum_{j \in M_i} b_{ij} = B_i \quad \text{for all } i \in N \\
& \sum_{i \in N_j} b_{ij} = x_j \quad \text{for all } j \in M \\
& x_j \leqslant \text{cap}_j \quad \text{for all } j \in M
\end{aligned}
$$

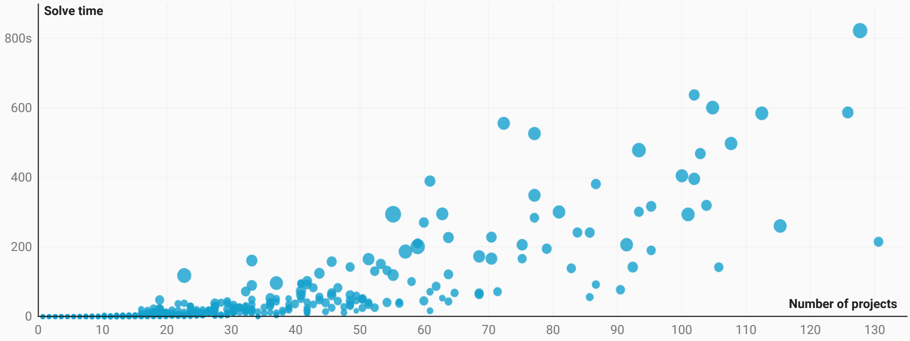
Theorem

An optimal solution to the above convex program forms a Lindahl equilibrium and thus lies in the core.

- The proof is straightforward by analyzing the KKT conditions of the program.

Capped public goods setting: Computation

- Our program can be solved efficiently (approximately) using many convex programming solvers with support for exponential cones (for example MOSEK).
- Can try it in your browser: `https://dominik-peters.de/demos/lindahl/`
- Fast even on large PB instances from pabulib, using MOSEK.



Questions for the future

- Is the optimum of our program unique in utilities?
  - Lindahl equilibrium is not unique under cap constraints, but our program might be.
  - On examples, it looks like our program selects particularly “fair” Lindahl equilibria. Can this be formalized?
- Can we develop first-order methods for the capped setting? The dynamics doesn’t seem to adapt easily.
- Can we generalize the cap constraints? This might allow for multi-issue decision making models.
- Can we connect private and public goods? Can we demystify why the results in the two models are so analogous?

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