The Core of Approval-Based Committee Elections with Few Seats

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Approval-Based Committee Elections

Task: select a *committee* $W \subseteq C$ consisting of k out of m *candidates*

Input: *approval preferences* (voters say which candidates they like). Voter *utility* is the number of approved candidates in the committee.

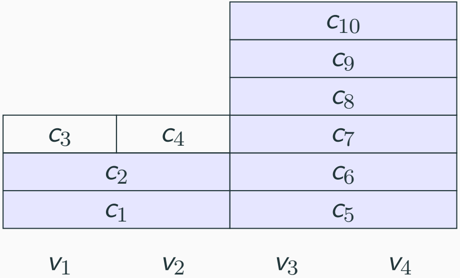
Aim: committee W should be representative and stable in the sense of the **core**:

> *a group of $x\%$ of voters cannot identify a small committee T with $\lfloor x\% \cdot k \rfloor$ members such that every group member strictly prefers the proposed small committee to W.*

Approval-Based Committee Elections

Example:

choose $k = 8$ candidates



The blue committee W fails the core because the group $S = \{v_1, v_2\}$ makes up 50% of the voters and can propose a small committee $T = \{c_1, c_2, c_3, c_4\}$ that they all prefer (they each approve 3 members of T but only 2 members of W).

Open Problem: Is the Core Always Non-Empty?

Aziz et al. [2016/2017] defined the core for approval-based committee elections. They noted that all known voting rules fail the core. They left open whether there always exists a core-stable committee. This **remains open**.

> Haris Aziz, Markus Brill, Vincent Conitzer, Edith Elkind, Rupert Freeman, and Toby Walsh. "Justified representation in approval-based committee voting". In: *Social Choice and Welfare* 48.2 (2017), pp. 461–485. DOI: 10.1007/s00355-016-1019-3

Trying to search for counterexamples to existence by computer is difficult: need to impose constraints that $\binom{m}{k}$ committees all fail the core due to one of $O(2^m)$ small committees. ILPs stop working quickly. (Gurobi solves $m = 7$, $k = 5$, in 450s, but did not solve $m = 7$, $k = 4$ after 134 000s (37h) on 8 cores.)

Open Problem: Is the Core Always Non-Empty?

Angles of attack:

- Relaxations: very well studied
    - Extended Justified Representation (EJR) requires that the group of voters is cohesive (everyone in the group approves all of T). This is satisfied by several natural rules (PAV, Method of Equal Shares, among others).

> Haris Aziz, Markus Brill, Vincent Conitzer, Edith Elkind, Rupert Freeman, and Toby Walsh. "Justified representation in approval-based committee voting". In: *Social Choice and Welfare* 48.2 (2017), pp. 461–485. DOI: 10.1007/s00355-016-1019-3

    - Full Justified Representation (FJR) is an even milder relaxation and still satisfiable.

> Dominik Peters, Grzegorz Pierczyński, and Piotr Skowron. "Proportional participatory budgeting with additive utilities". In: *Advances in Neural Information Processing Systems*. Vol. 34. 2021, pp. 12726–12737. DOI: 10.48550/arXiv.2008.13276

    - Only allow deviations that make everyone in S better off in a superset sense ($A_i \cap T \supsetneq A_i \cap W$): satisfied by Phragmén.

> Kamesh Munagala, Yiheng Shen, and Kangning Wang. "Auditing for core stability in participatory budgeting". In: *Proceedings of the 18th International Conference on Web and Internet Economics (WINE)*. Springer. 2022, pp. 292–310. DOI: 10.1007/978-3-031-22832-2_17

Open Problem: Is the Core Always Non-Empty?

Angles of attack:

- Approximations: very well studied (with improvements happening rapidly right now)
  - Require coalition to be larger by a factor of $\beta \geqslant 1$. Existence for $\beta = 16$ (in 2020) $\to 4.97$ (February 2025) $\to 3.65$ (August 2025).

> Zhihao Jiang, Kamesh Munagala, and Kangning Wang. "Approximately stable committee selection". In: *Proceedings of the 52nd Annual ACM SIGACT Symposium on Theory of Computing (STOC)*. 2020, pp. 463–472. DOI: 10.1145/3357713.3384238

> Thàn Nguyen and Haoyu Song. "Approximate Core of Participatory Budgeting via Lindahl Equilibrium". In: (2025). Working Paper

> Drew Gao, Yihang Sun, and Jan Vondrák. *Computation of approximately stable committees in approval-based elections*. 2025. arXiv: 2508.00130 [cs.GT]. URL: https://arxiv.org/abs/2508.00130

  - Require coalition to be happier by a factor of $\alpha \geqslant 1$. Existence for $\alpha = \log k$ (MES rule), $\alpha = 2$ (PAV rule), $\alpha = 9.27$ (additive valuations).

> Dominik Peters and Piotr Skowron. "Proportionality and the limits of welfarism". In: *Proceedings of the 21st ACM Conference on Economics and Computation (EC)*. Full version arXiv:1911.11747. 2020, pp. 793–794

> Kamesh Munagala, Yiheng Shen, Kangning Wang, and Zhiyi Wang. "Approximate core for committee selection via multilinear extension and market clearing". In: *Proceedings of the 2022 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*. 2022, pp. 2229–2252. DOI: 10.1137/1.9781611977073.89

Open Problem: Is the Core Always Non-Empty?

Angles of attack:

- Special cases
  - Committee size $k \leqslant 3$ (case analysis).

> Yu Cheng, Zhihao Jiang, Kamesh Munagala, and Kangning Wang. "Group fairness in committee selection". In: *ACM Transactions on Economics and Computation (TEAC)* 8.4 (2020), pp. 1–18. DOI: 10.1145/3417750

  - Candidates each have k copies.

> Markus Brill, Paul Gölz, Dominik Peters, Ulrike Schmidt-Kraepelin, and Kai Wilker. "Approval-based apportionment". In: *Mathematical Programming* (2022). DOI: 10.1007/s10107-022-01852-1

  - Single-peaked profiles (candidate or voter interval).

> Grzegorz Pierczyński and Piotr Skowron. "Core-stable committees under restricted domains". In: *Proceedings of the 18th International Conference on Web and Internet Economics (WINE)*. Springer. 2022, pp. 311–329. DOI: 10.1007/978-3-031-22832-2_18

Result 1: Core Exists for $k \leqslant 8$ Seats

Thiele [1895] proposed a voting rule now called *Proportional Approval Voting* (PAV). It selects those committees that maximize

$$\sum_{\text{voters } i} 1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \frac{1}{u_i(W)} \quad \text{where } u_i(W) = |A_i \cap W|.$$

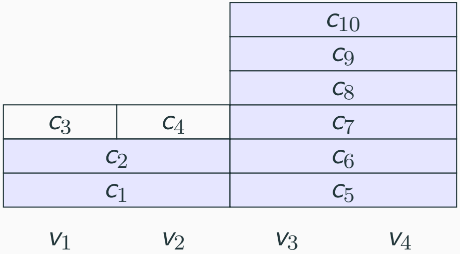
Thorvald N. Thiele. “Om Flerfold Valg”. In: *Oversigt over det Kongelige Danske Videnskabernes Selskabs Fordhandlinger* (1895). URL: https://dominik-peters.de/archive/thiele1895.pdf

Theorem. For $k \leqslant 7$, every PAV committee is in the core.

Proof by solving linear programs looking for profiles where $W = \{c_1, \dots, c_k\}$ is selected by PAV but fails the core due to some T with too much support. (Iterate over all candidates for T up to symmetries.) Results are certified via Farkas lemma.

$\implies$ core is non-empty for $k \leqslant 7$ (for any number of voters/candidates)

Result 1: Core Exists for $k \leqslant 8$ Seats



PAV may fail the core for $k = 8$ (the blue committee is selected by PAV). But there are other (tied) committees selected by PAV that are in the core (add c_4, remove c_{10}). This generalizes:

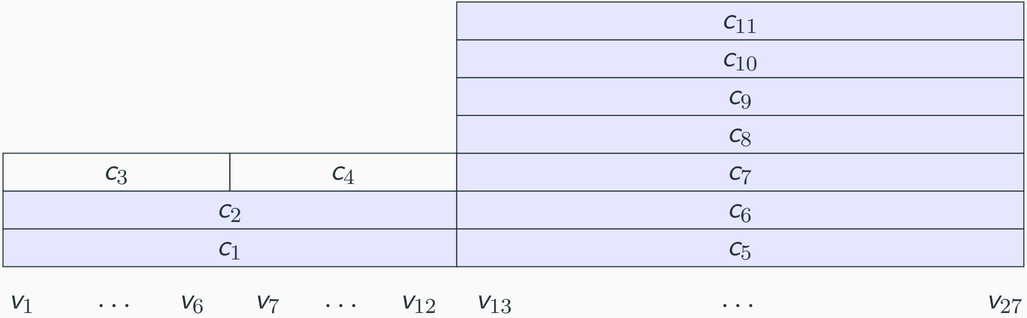
Theorem. For $k = 8$, at least one PAV committee is in the core.

Proof by linear programs showing that the above example is essentially the only example where PAV fails the core for $k = 8$.

$\Longrightarrow$ core is non-empty for $k \leqslant 8$ (for any number of voters/candidates)

Result 1: Core Exists for $k \leqslant 8$ Seats

For $k = 9$, there are examples where *all* PAV committees fail the core.



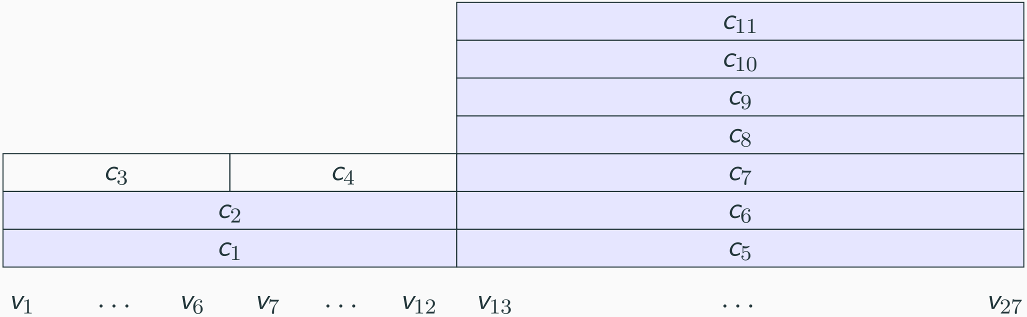
Result 2: Core Exists for $m \leqslant 15$ Candidates

So our technique stops working for $k \geqslant 9$.

Can we get existence results for small m (number of candidates)?

Result 2: Core Exists for $m \leqslant 15$ Candidates

Note that the counterexamples for PAV are very "nice" (they are even laminar!). Thought: maybe we can "fix" them.



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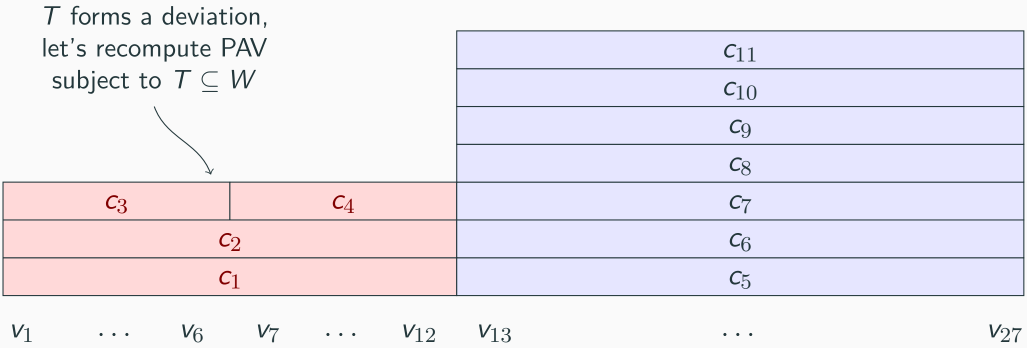

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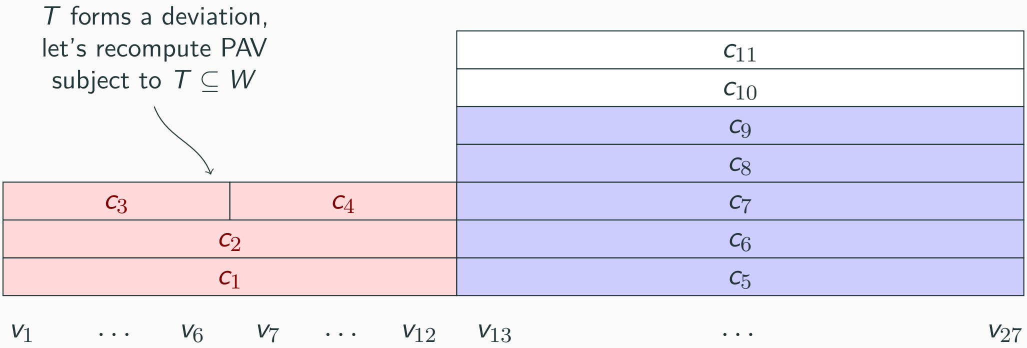

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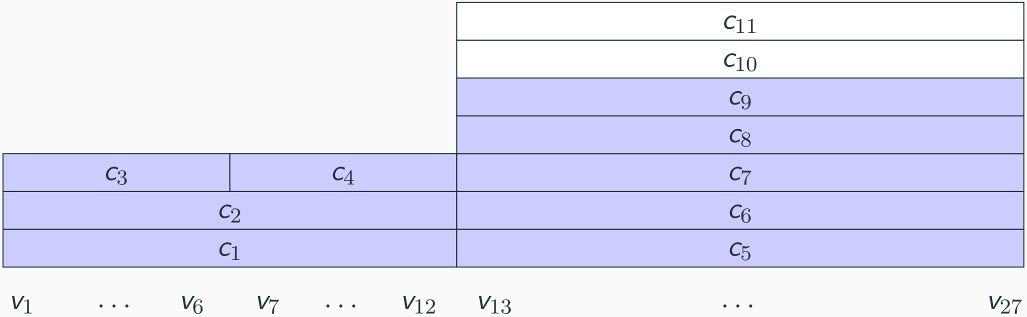

✓ in the core

Result 2: Core Exists for $m \leqslant 15$ Candidates

We define a *recursive PAV rule* and prove that it selects core committees whenever the number of candidates is at most 15.

The rule computes PAV, and checks if the result is in the core. If not, find deviation T, and re-compute PAV subject to the constraint $T \subseteq W$ (ignoring voters who were part of the deviation). Repeat and add further constraints if necessary.

Proof using LPs, with Farkas certified results.

Recursive PAV rule stops working for $m = 16$ and $k \in \{10, 11\}$.

Remarks and Future Directions

- Core is defined using the *Hare quota*, where a group deserves ℓ candidates if it has size $\geqslant \ell \cdot \frac{n}{k}$. We can define a stronger version using the *Droop quota* which applies to groups of size $> \ell \cdot \frac{n}{k+1}$. But for Droop quota we don't even know existence for $k = 6$ and $m = 10$.
- The Method of Equal Shares (MES) and Phragmén's method fail the core even for $k = 7$ and $k = 6$, resp. But it isn't possible to find such counterexamples using LPs [Xia 2025].
- We do not have examples where PAV and MES/Phragmén fail the core *simultaneously*. We do not have examples where the rule maximizing PAV score subject to *priceability* fails the core.
- All results hold even for "local swap" versions of PAV.
- Core existence is open even when the "unit cost" assumption is dropped (i.e. replace cardinality constraint by a knapsack constraint).

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