Approval-Based Apportionment

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Abstract

In the apportionment problem, a fixed number of seats must be distributed among parties in proportion to the number of voters supporting each party. We study a generalization of this setting, in which voters cast approval ballots over parties, such that each voter can support multiple parties. This approval-based apportionment setting generalizes traditional apportionment and is a natural restriction of approval-based multiwinner elections, where approval ballots range over individual candidates. Using techniques from both apportionment and multiwinner elections, we are able to provide representation guarantees that are currently out of reach in the general setting of multiwinner elections: First, we show that core-stable committees are guaranteed to exist and can be found in polynomial time. Second, we demonstrate that extended justified representation is compatible with committee monotonicity.

1 Introduction

The fundamental fairness principle of *proportional representation* is relevant in a variety of applications ranging from recommender systems to digital democracy (Skowron *et al.* 2017). It features most explicitly in the context of political elections, which is the language we adopt for this paper. In this context, proportional representation prescribes that the number of representatives championing a particular opinion in a legislature be proportional to the number of voters who favor that opinion.

In most democratic institutions, proportional representation is implemented via *party-list elections*: Candidates are members of political parties and voters are asked to indicate their favorite party; each party is then allocated a number of seats that is (approximately) proportional to the number of votes it received. The problem of transforming a voting outcome into a distribution of seats is known as *apportionment*. Analyzing the advantages and disadvantages of different apportionment methods has a long and illustrious political history and has given rise to a deep and elegant mathematical theory (Balinski and Young 1982; Pukelsheim 2014).

Unfortunately, forcing voters to choose a single party prevents them from communicating any preferences beyond their most preferred alternative. For example, if a voter feels equally well represented by several political parties, there is no way to express this preference within the voting system.

In the context of single-winner elections, *approval voting* has been put forward as a solution to this problem as it strikes an attractive compromise between simplicity and expressivity (Brams and Fishburn 2007; Laslier and Sanver 2010). Under approval voting, each voter is asked to specify a set of candidates she "approves of," i.e., voters can arbitrarily partition the set of candidates into approved candidates and disapproved ones. Proponents of approval voting argue that its introduction could increase voter turnout, "help elect the strongest candidate," and "add legitimacy to the outcome" of an election (Brams and Fishburn 2007, pp. 4–8).

Due to the practical and theoretical appeal of approval voting in single-winner elections, a number of scholars have suggested to also use approval voting for multiwinner elections, in which a fixed number of candidates needs to be elected (Kilgour and Marshall 2012). In contrast to the single-winner setting, where the straightforward voting rule "choose the candidate approved by the highest number of voters" enjoys a strong axiomatic foundation (Fishburn 1978), several ways of aggregating approval ballots have been proposed in the multiwinner setting (e.g., Aziz *et al.* 2017; Janson 2016).

Most studies of approval-based multiwinner elections assume that voters directly express their preference over individual candidates; we refer to this setting as *candidate-approval* elections. This assumption runs counter to widespread democratic practice, in which candidates belong to political parties and voters indicate preferences over these parties (which induce implicit preferences over candidates). In this paper, we therefore study *party-approval* elections, in which voters express approval votes over parties and a given number of seats must be distributed among the parties. We refer to the process of allocating these seats as *approval-based apportionment*.

We believe that party-approval elections are a promising framework for legislative elections in the real world. Allowing voters to express approval votes over parties enables the aggregation mechanism to coordinate like-minded voters. For example, two blocks of voters might currently vote for parties that they mutually disapprove of. Using approval ballots could reveal that the blocks jointly approve a party of more general appeal; allocating more seats to this party leads to mutual gain. This cooperation is particularly necessary for small minority opinions that are not centrally coordinated. In such cases, finding a commonly approved party can make the difference between being represented or votes being wasted because the individual parties receive insufficient support.

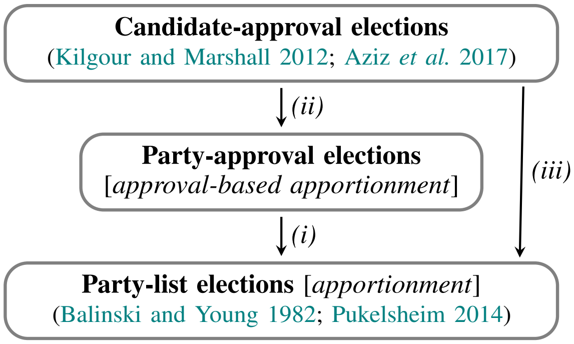

Figure 1: Relations between the different settings of multiwinner elections. An arrow from X to Y signifies that X is a generalization of Y. The relationship corresponding to arrow *(iii)* has been explored by Brill *et al.* (2018). We establish and explore the relationship *(i)* in Section 3 and the relationship *(ii)* in Section 4.

In contrast to approval voting over individual candidates, party-approval voting does not require a break with the current role of political parties—it can be combined with both "open list" and "closed list" approaches to filling the seats allocated to a party.

1.1 Related Work

To the best of our knowledge, this paper is the first to formally develop and systematically study approval-based apportionment. That is not to say that the idea of expressing and aggregating approval votes over parties has not been considered before. Indeed, several scholars have explored possible generalizations of existing aggregation procedures.

For instance, Brams *et al.* (2019) study multiwinner approval rules that are inspired by classical apportionment methods. Besides the setting of candidate approval, they explicitly consider the case where voters cast party-approval votes. They conclude that these rules could "encourage coalitions across party or factional lines, thereby diminishing gridlock and promoting consensus."

Such desire for compromise is only one motivation for considering party-approval elections, as exemplified by recent work by Speroni di Fenizio and Gewurz (2019). To allow for more efficient governing, they aim to concentrate the power of a legislature in the hands of few big parties, while nonetheless preserving the principle of proportional representation. To this end, they let voters cast party-approval votes and transform these votes into a party-list election by assigning each voter to one of her approved parties. One method for doing this (referred to as *majoritarian portioning* later in this paper) assigns voters to parties in such a way that the strongest party has as many votes as possible.

Several other papers consider extensions of approval-based voting rules to accommodate party-approval elections (Brams and Kilgour 2014; Mora and Oliver 2015; Janson 2016; Janson and Öberg 2019). All of these papers have in common that they study specific rules or classes of rules, rather than exploring the party-approval setting in its own right.

1.2 Relation to Other Settings

Party-approval elections can be positioned between two well-studied voting settings (see Figure 1).

First, approval-based apportionment *generalizes standard apportionment*, which corresponds to party-approval elections in which all approval sets are singletons. This relation (depicted as arrow *(i)* in Figure 1) provides a generic two-step approach to define aggregation rules for approval-based apportionment problems: transform a party-approval instance to an apportionment instance, and then apply an apportionment method. In Section 3, we employ this approach to construct approval-based apportionment methods satisfying desirable properties.

Second, our setting can be viewed as a *special case of approval-based multiwinner voting*, in which voters cast *candidate-approval* votes. A party-approval election can be embedded in this setting by replacing each party by multiple candidates belonging to this party, and by interpreting a voter's approval of a party as approval of all of its candidates. This embedding establishes party-approval elections as a subdomain of candidate-approval elections (see arrow *(ii)* in Figure 1). In Section 4, we explore the axiomatic and computational ramifications of this domain restriction.

1.3 Contributions

In this paper, we formally introduce the setting of approval-based apportionment and explore different possibilities of constructing axiomatically desirable aggregation methods for this setting. Besides its conceptual appeal, this setting is also interesting from a technical perspective.

Exploiting the relations described in Section 1.2, we resolve problems that remain open in the more general setting of approval-based multiwinner voting. First, we prove that committee monotonicity is compatible with extended justified representation (a representation axiom proposed by Aziz *et al.* 2017) by providing a rule that satisfies both properties. Second, we show that the core of an approval-based apportionment problem is always nonempty and that core-stable committees can be found in polynomial time.

Besides these positive results, we verify for a wide range of multiwinner voting rules that their axiomatic guarantees do not improve in the party-approval setting, and that some rules remain NP-hard to evaluate. On the other hand, we show that it becomes tractable to check whether a committee provides extended justified representation or the weaker axiom of proportional justified representation.

Omitted proofs as well as further definitions and results can be found in Appendices A and B.

2 The Model

A *party-approval election* is a tuple (N, P, A, k) consisting of a set of voters $N = \{1, \dots, n\}$, a finite set of parties P, a ballot profile $A = (A_1, \dots, A_n)$ where each ballot $A_i \subseteq P$ is the set of parties approved by voter i, and the committee size $k \in \mathbb{N}$. We assume that $A_i \neq \emptyset$ for all $i \in N$. When considering computational problems, we assume that k is encoded in unary (see Footnote 5).

A *committee* in this setting is a multiset $W : P \to \mathbb{N}$ over parties, which determines the number of seats $W(p)$ assigned to each party $p \in P$. The size of a committee W is given by $|W| = \sum_{p\in P} W(p)$, and we denote multiset addition and subtraction by $+$ and $-$, respectively. A *party-approval rule* is a function that takes a party-approval election (N, P, A, k) as input and returns a committee W of valid size $|W| = k$.[1]

In our axiomatic study of party-approval rules, we focus on two axioms capturing proportional representation: extended justified representation and core stability (Aziz *et al.* 2017).[2] Both axioms are derived from their analogs in multiwinner elections (see Section 4.2) and can be defined in terms of quota requirements.

For a party-approval election (N, P, A, k) and a subset $S \subseteq N$ of voters, define the *quota of* S as $q(S) = \lfloor k \cdot |S|/n \rfloor$. Intuitively, $q(S)$ corresponds to the number of seats that the group S "deserves" to be represented by (rounded down).

Definition 1. *A committee $W : P \to \mathbb{N}$ provides* extended justified representation (EJR) *for a party-approval election (N, P, A, k) if there is no subset $S \subseteq N$ of voters such that $\bigcap_{i\in S} A_i \neq \emptyset$ and $\sum_{p\in A_i} W(p) < q(S)$ for all $i \in S$.*

In words, EJR requires that for every voter group S with a commonly approved party, at least one voter of the group should be represented by $q(S)$ many candidates. A party-approval rule is said to *satisfy EJR* if it only produces committees providing EJR.

We can obtain a stronger representation axiom by removing the requirement of a commonly approved party.

Definition 2. *A committee $W : P \to \mathbb{N}$ is* core stable *for a party-approval election (N, P, A, k) if there is no nonempty subset $S \subseteq N$ and committee $T : P \to \mathbb{N}$ of size $|T| \leqslant q(S)$ such that $\sum_{p\in A_i} T(p) > \sum_{p\in A_i} W(p)$ for all $i \in S$. The* core *of a party-approval election is defined as the set of all core-stable committees.*

Core stability requires adequate representation even for voter groups that cannot agree on a common party, by ruling out the possibility that the group can deviate to a smaller committee that represents all voters in the group strictly better. It follows from the definitions that core stability is a stronger requirement than EJR: If a committee violates EJR, there is a group S that would prefer any committee of size $q(S)$ that assigns all seats to the commonly approved party.

A final, non-representational axiom that we will discuss is *committee monotonicity*. A party-approval rule f satisfies this axiom if, for all party-approval elections (N, P, A, k), it holds that $f(N, P, A, k) \subseteq f(N, P, A, k + 1)$. Committee monotonic rules avoid the so-called *Alabama paradox*, in which a party loses a seat when the committee size increases. Besides, committee monotonic rules can be used to construct proportional rankings (Skowron *et al.* 2017).

[1] This definition implies that rules are *resolute*, that is, only a single committee is returned. In the case of a tie between multiple committees, a tiebreaking mechanism is necessary. Our results hold independently of the choice of a specific tiebreaking mechanism.

[2] Some results in the appendix refer to the weaker representation axioms of *justified representation (JR)* (Aziz *et al.* 2017) and *proportional justified representation (PJR)* (Sánchez-Fernández *et al.* 2017); see Appendix A.1. It is well known that EJR implies PJR and that PJR implies JR.

3 Constructing Party-Approval Rules via Portioning and Apportionment

Party-approval elections are a generalization of party-list elections, which can be thought of as party-approval elections in which all approval sets are singletons. Since there is a rich body of research on apportionment methods, it is natural to examine whether we can employ these methods for our setting as well. To use them, we will need to translate party-approval elections into the party-list domain on which apportionment methods operate. This translation thus needs to transform a collection of approval votes over parties into vote shares for each party. Motivated by time sharing, Bogomolnaia *et al.* (2005) have developed a theory of such transformation rules, further studied by Duddy (2015) and Aziz *et al.* (2019). We will refer to this framework as *portioning*.

The approach explored in this section, then, divides the construction of a party-approval rule into two independent steps: (1) portioning, which maps a party-approval election to a vector of parties' shares; followed by (2) apportionment, which transforms the shares into a seat distribution.

Both the portioning and the apportionment literature have discussed representation axioms similar in spirit to EJR and core stability. For both settings, several rules have been found to satisfy these properties. One might hope that by composing two rules that are each representative, we obtain a party-approval rule that is also representative (and satisfies, say, EJR). If we succeed in finding such a combination, it is likely that the resulting voting rule will automatically satisfy committee monotonicity since most apportionment methods satisfy this property. In the general candidate-approval setting (considered in Section 4), the existence of a rule satisfying both EJR and committee monotonicity is an open problem.

3.1 Preliminaries

We start by introducing relevant notions from the literature of portioning (Bogomolnaia *et al.* 2005; Aziz *et al.* 2019) and apportionment (Balinski and Young 1982; Pukelsheim 2014), with notations and interpretations suitably adjusted to our setting.

Portioning A *portioning problem* is a triple (N, P, A), just as in party-approval voting but without a committee size. A *portioning* is a function $r : P \to [0, 1]$ with $\sum_{p\in P} r(p) = 1$. We interpret $r(p)$ as the vote share of party p. A *portioning method* maps each triple (N, P, A) to a portioning.

Our minimum requirement on portioning methods will be that they uphold proportionality if all approval sets are singletons, i.e., if we are already in the party-list domain. Formally, we say that a portioning method is *faithful* if for all (N, P, A) with $|A_i| = 1$ for all $i \in N$, the resulting portioning r satisfies $r(p) = |\{i \in N \mid A_i = \{p\}\}|/n$ for all $p \in P$. Among the portioning methods considered by Aziz *et al.* (2019), only three are faithful. They are defined as follows.

Conditional utilitarian portioning selects, for each voter i, p_i as a party in A_i approved by the highest number of voters. Then, $r(p) = |\{i \in N \mid p_i = p\}|/n$ for all $p \in P$.

Random priority computes $n!$ portionings, one for each permutation σ of N, and returns their average. The portioning for $\sigma = (i_1, \dots, i_n)$ maximizes $\sum_{p \in A_{i_1}} r(p)$, breaking ties by maximizing $\sum_{p \in A_{i_2}} r(p)$, and so forth.

Nash portioning selects the portioning r maximizing the Nash welfare $\prod_{i \in N} \big(\sum_{p \in A_i} r(p) \big)$.

The last method seems particularly promising because it satisfies portioning versions of core stability and EJR (Aziz *et al.* 2019).

We will also make use of a more recent portioning approach, which was proposed by Speroni di Fenizio and Gewurz (2019) in the context of party-approval voting.

Majoritarian portioning proceeds in rounds $j = 1, 2, \dots$. Initially, all parties and voters are *active*. In iteration j, we select the active party p_j that is approved by the highest number of active voters. Let N_j be the set of active voters who approve p_j. Then, set $r(p_j)$ to $|N_j|/n$, and mark p_j and all voters in N_j as inactive. If active voters remain, the next iteration is started; else, r is returned.

Under majoritarian portioning, the approval preferences of voters who have been assigned to a party are ignored in further iterations. Note that conditional utilitarian portioning can similarly be seen as a sequential method, in which the preferences of inactive voters are not ignored.

Apportionment An *apportionment problem* is a tuple (P, r, k), which consists of a finite set of parties P, a portioning $r : P \to [0, 1]$ specifying the vote shares of parties, and a committee size $k \in \mathbb{N}$. Committees are defined as for party-approval elections, and an *apportionment method* maps apportionment problems to committees W of size k.

An apportionment method satisfies *lower quota* if each party p is always allocated at least $\lfloor k \cdot r(p) \rfloor$ seats in the committee. Furthermore, an apportionment method f is *committee monotonic* if $f(P, r, k) \subseteq f(P, r, k+1)$ for every apportionment problem (P, r, k).

Among the standard apportionment methods, only one satisfies both lower quota and committee monotonicity: the *D'Hondt method* (aka *Jefferson method*).[3] The method assigns the k seats iteratively, each time giving the next seat to the party p with the largest quotient $r(p)/(s(p)+1)$, where $s(p)$ denotes the number of seats already assigned to p. Another apportionment method satisfying lower quota and committee monotonicity is the *quota method*, due to Balinski and Young (1975). It is identical to the D'Hondt method, except that, in the jth iteration, only parties p satisfying $s(p)/j < r(p)$ are eligible for the allocation of the next seat.

Composition If we take any portioning method and any apportionment method, we can compose them to obtain a party-approval rule. Note that if the apportionment method is committee monotonic then so is the composed rule, since the portioning is independent of k.

[3] All other *divisor methods* fail lower quota, and the *Hamilton method* is not committee monotonic (Balinski and Young 1982).

3.2 Composed Rules That Fail EJR

Perhaps surprisingly, many pairs of portioning and apportionment methods fail EJR. This is certainly true if the individual parts are not representative themselves. For example, if an apportionment method M properly fails lower quota (in the sense that there is a rational-valued input r on which lower quota is violated), then one can construct an example profile on which any composed rule using M fails EJR: Construct a party-approval election with singleton approval sets in which the voter counts are proportional to the shares in the counterexample r. Then any faithful portioning method, applied to this election, must return r. Since M fails lower quota on r, the resulting committee will violate EJR. By a similar argument, an apportionment method that violates committee monotonicity on some rational portioning will, when composed with a faithful portioning method, give rise to a party-approval rule that fails committee monotonicity.

To our knowledge, among the named and studied apportionment methods, only two satisfy both lower quota and committee monotonicity: D'Hondt and the quota method. However, it turns out that the composition of either option with the conditional-utilitarian, random-priority, or Nash portioning methods fails EJR, as the following examples show.

Example 1. *Let* $n = k = 6$*,* $P = \{p_0, p_1, p_2, p_3\}$*,* $A = (\{p_0\}, \{p_0\}, \{p_0, p_1, p_2\}, \{p_0, p_1, p_2\}, \{p_1, p_3\}, \{p_2, p_3\})$*.*

Then, the conditional utilitarian solution sets $r(p_0) = 4/6$*,* $r(p_1) = r(p_2) = 1/6$*, and* $r(p_3) = 0$*. Any apportionment method satisfying lower quota allocates four seats to* p_0*, one each to* p_1 *and* p_2*, and none to* p_3*. The resulting committee does not provide EJR since the last two voters, who jointly approve* p_3*, have a quota of* $q(\{5, 6\}) = 2$ *that is not met.*

Example 2. *Let* $n = k = 6$*,* $P = \{p_0, p_1, p_2, p_3\}$*, and* $A = (\{p_0\}, \{p_0\}, \{p_0, p_1, p_2\}, \{p_0, p_1, p_3\}, \{p_1\}, \{p_2, p_3\})$*.*

Random priority chooses the portioning $r(p_0) = 23/45$*,* $r(p_1) = 23/90$*, and* $r(p_2) = r(p_3) = 7/60$*. Both D'Hondt and the quota method allocate four seats to* p_0*, two seats to* p_1*, and none to the other two parties. This clearly violates the claim to representation of the sixth voter (with* $q(\{6\}) = 1$*).*

Nash portioning produces a fairly similar portioning, with $r(p_0) \approx 0.5302$*,* $r(p_1) \approx 0.2651$*, and* $r(p_2) = r(p_3) \approx 0.1023$*. D'Hondt and the quota method produce the same committee as above, leading to the same EJR violation.*

At first glance, it might be surprising that Nash portioning combined with a lower-quota apportionment method violates EJR (and even the weaker axiom JR). Indeed, Nash portioning satisfies core stability in the portioning setting, which is a strong notion of proportionality, and the lower-quota property limits the rounding losses when moving from the portioning to a committee. As expected, in the election of Example 2, the Nash solution itself gives sufficient representation to the sixth voter since $r(p_2) + r(p_3) \approx 0.2047 > 1/6$. However, since both $r(p_2)$ and $r(p_3)$ are below $1/6$ on their own, lower quota does not apply to either of the two parties, and the sixth voter loses all representation in the apportionment step.

3.3 Composed Rules That Satisfy EJR

As we have seen, several initially promising portioning methods fail to compose to a rule that satisfies EJR. One reason is that these portioning methods are happy to assign small shares to several parties. The apportionment method may round several of those small shares down to zero seats. This can lead to a failure of EJR when not enough parties obtain a seat. It is difficult for an apportionment method to avoid this behavior since the portioning step obscures the relationships between different parties that are apparent from the approval ballots of the voters.

Majoritarian portioning is designed to maximize the seat allocations to the largest parties. Thus, it tends to avoid the problem we have identified. While it fails the strong representation axioms that Nash portioning satisfies, this turns out not to be crucial: Composing majoritarian portioning with any apportionment method satisfying lower quota yields an EJR rule. If we use an apportionment method that is also committee monotonic, such as D'Hondt or the quota method, we obtain a party-approval rule that satisfies both EJR and committee monotonicity.[4]

Theorem 1. *Let M be a committee monotonic apportionment method satisfying lower quota. Then, the party-approval rule composing majoritarian portioning and M satisfies EJR and committee monotonicity.*

Proof. Consider a party-approval election (N, P, A, k) and let r be the outcome of majoritarian portioning applied to (N, P, A). Let $N_1, N_2, \ldots$ and $p_1, p_2, \ldots$ be the voter groups and parties in the construction of majoritarian portioning, so that $r(p_j) = |N_j|/n$ for all j.

Consider the committee $W = M(P, r, k)$ and suppose that EJR is violated, i.e., that there exists a group $S \subseteq N$ with $\bigcap_{i \in S} A_i \neq \emptyset$ and $\sum_{p \in A_i} W(p) < q(S)$ for all $i \in S$.

Let j be minimal such that $S \cap N_j \neq \emptyset$. We now show that $|S| \leqslant |N_j|$. By the definition of j, no voter in S approves of any of the parties $p_1, p_2, \ldots p_{j-1}$; thus, all those voters remain active in round j. Consider a party $p^* \in \bigcap_{i \in S} A_i$. In the jth iteration of majoritarian portioning, this party had an approval score of at least $|S|$. Therefore, the party p_j that is chosen in the jth iteration has an approval score that is at least $|S|$ (of course, $p^* = p_j$ is possible). The approval score of party p_j equals $|N_j|$. Therefore, $|N_j| \geqslant |S|$.

Since $|N_j| \geqslant |S|$, we have $q(N_j) \geqslant q(S)$. Since M satisfies lower quota, it assigns at least $\lfloor k \cdot r(p_j) \rfloor = \lfloor k(|N_j|/n) \rfloor = q(N_j)$ seats to party p_j. Now consider a voter $i \in S \cap N_j$. Since this voter approves party p_j, we have $\sum_{p \in A_i} W(p) \geqslant W(p_j) \geqslant q(N_j) \geqslant q(S)$, a contradiction.

This shows that EJR is indeed satisfied; committee monotonicity follows from the committee monotonicity of M. $\square$

While the party-approval rules identified by Theorem 1 satisfy EJR and committee monotonicity, they do not quite reach our gold standard of representation, i.e., core stability.

[4] As long as the apportionment method is computable in polynomial time (which is the case for D'Hondt and the quota method), the same holds for the resulting party-approval rule.

Example 3. *Let $n = k = 16$, let $P = \{p_0, \ldots, p_4\}$, with the following approval sets: 4 times $\{p_0, p_1\}$, 3 times $\{p_1, p_2\}$, once $\{p_2\}$, 4 times $\{p_0, p_3\}$, 3 times $\{p_3, p_4\}$, and once $\{p_4\}$. Note the symmetry between p_1 and p_3, and between p_2 and p_4. Majoritarian portioning allocates $1/2$ to p_0 and $1/4$ each to p_2 and p_4. Any lower-quota apportionment method must translate this into 8 seats for p_0 and 4 seats each for p_2 and p_4. This committee is not in the core: Let S be the coalition of all 14 voters who approve multiple parties, and let T allocate 4 seats to p_0 and 5 seats each to p_1 and p_3. This gives strictly higher representation to all members of the coalition.*

The example makes it obvious why majoritarian portioning cannot satisfy the core: All voters approving of p_0 get deactivated after the first round, which makes p_2 seem universally preferable to p_1. However, p_1 is a useful vehicle for cooperation between the group approving $\{p_0, p_1\}$ and the group approving $\{p_1, p_2\}$. Since majoritarian portioning is blind to this opportunity, it cannot guarantee core stability.

The example also illustrates the power of core stability: The deviating coalition does not agree on any single party they support, but would nonetheless benefit from the deviation. There is room for collaboration, and core stability is sensitive to this demand for better representation.

4 Constructing Party-Approval Rules via Multiwinner Voting Rules

In the previous section, we applied tools from apportionment, a more restrictive setting, to our party-approval setting. Now, we go in the other direction, and apply tools from a more general setting: As mentioned in Section 1.2, party-approval elections can be viewed as a special case of candidate-approval elections, i.e., multiwinner elections in which approvals are expressed over individual candidates rather than parties. After introducing relevant candidate-approval notions, we show how party-approval elections can be translated into candidate-approval elections. This embedding allows us to apply established candidate-approval rules to our setting. Exploiting this fact, we will prove the existence of core-stable committees for party-approval elections.

4.1 Preliminaries

A *candidate-approval election* is a tuple (N, C, A, k). Just as for party-approval elections, $N = \{1, \ldots, n\}$ is a set of voters, C is a finite set, A is an n-tuple of nonempty subsets of C, and $k \in \mathbb{N}$ is the committee size. The conceptual difference is that C is a set of individual candidates rather than parties. This difference manifests itself in the definition of a committee because a single candidate cannot receive multiple seats. That is, a *candidate committee* W is now simply a subset of C with cardinality k. (Therefore, it is usually assumed that $|C| \geqslant k$.) A *candidate-approval rule* is a function that maps each candidate-approval election to a candidate committee.

A diverse set of such voting rules has been proposed since the late 19th century (Kilgour and Marshall 2012; Janson 2016; Aziz *et al.* 2017), out of which we will only introduce the one which we use for our main positive result. Let H_j denote the jth harmonic number, i.e., $H_j = \sum_{t=1}^{j} 1/t$.

Given (N, C, A, k), the candidate-approval rule *proportional approval voting (PAV)*, introduced by Thiele (1895), chooses a candidate committee W maximizing the PAV score $\text{PAV}(W) = \sum_{i \in N} H_{|W \cap A_i|}$.

We now describe EJR and core stability in the candidate-approval setting, from which our versions of these axioms are derived. Recall that $q(S) = \lfloor k\,|S|/n \rfloor$. A candidate committee W provides ***EJR*** if there is no subset $S \subseteq N$ and no integer $\ell > 0$ such that $q(S) \geqslant \ell$, $|\bigcap_{i \in S} A_i| \geqslant \ell$, and $|A_i \cap W| < \ell$ for all $i \in S$. (The requirement $|\bigcap_{i \in S} A_i| \geqslant \ell$ is often referred to as *cohesiveness*.) A candidate-approval rule satisfies EJR if it always produces EJR committees.

The definition of core stability is even closer to the version in party-approval: A candidate committee W is *core stable* if there is no nonempty group $S \subseteq N$ and no set $T \subseteq C$ of size $|T| \leqslant q(S)$ such that $|A_i \cap T| > |A_i \cap W|$ for all $i \in S$. The *core* consists of all core-stable candidate committees.

4.2 Embedding Party-Approval Elections

We have informally argued in Section 1.2 that party-approval elections constitute a subdomain of candidate-approval elections. We formalize this notion by providing an embedding of party-approval elections into the candidate-approval domain. For a given party-approval election (N, P, A, k), we define a corresponding candidate-approval election with the same set of voters N and the same committee size k. The set of candidates contains k many "clone" candidates $p^{(1)}, \ldots, p^{(k)}$ for each party $p \in P$, and a voter approves a candidate $p^{(j)}$ in the candidate-approval election iff she approves the corresponding party p in the party-approval election. This embedding establishes party-approval elections as a subdomain of candidate-approval elections. As a consequence, we can apply rules and axioms from the more general candidate-approval setting also in the party-approval setting.

In particular, the generic way to apply a candidate-approval rule for a party-approval election consists in (1) translating the party-approval election into a candidate-approval election, (2) applying the candidate-approval rule, and (3) counting the number of chosen clones per party to construct a committee over parties. Note that, since k is encoded in unary, the running time is blown up by at most a polynomial factor.[5]

Having established party-approval elections as a subdomain of candidate-approval elections, our variants of EJR and core stability (Definitions 1 and 2) are immediately induced by their candidate-approval counterparts. In particular, any candidate-approval rule satisfying an axiom in the candidate-approval setting will satisfy the corresponding axiom in the party-approval setting as well. Note that, by restricting our view to party approval, the cohesiveness requirement of EJR is reduced to requiring a single commonly approved party.

[5]For candidate-approval elections, it does not make sense to have more seats than candidates, whereas for party-approval elections it is natural to have more seats than parties. If k was encoded in binary, even greedy candidate-approval algorithms would suddenly have exponential running time. This would complicate running-time comparisons between the candidate-approval and party-approval setting and would blur the intuitive distinction between simple and complex algorithms. Encoding k in unary sidesteps this technical complication.

4.3 PAV Guarantees Core Stability

A powerful stability concept in economics, core stability is a natural extension of EJR. It is particularly attractive because blocking coalitions are not required to be coherent at all, just to be able to coordinate for mutual gain. Our earlier Example 3 illustrates how a coalition might deviate in spite of not agreeing on any approved party.

Unfortunately, it is still unknown whether core-stable candidate committees exist for all candidate-approval elections. Fain *et al.* (2018) give positive approximate results for a variant of core stability in which blocking coalitions $S \subseteq N$ get to provide sets of candidates T of size k but have to increase their utilities by at least a factor of $n/|S|$ to be counterexamples to their notion of core stability. They provide a nonconstant approximation to the core in our sense, but nonemptyness remains open. Recently, Cheng *et al.* (2019) showed that there always exist randomized committees providing core stability (over expected representation), but it is not clear how their approach based on two-player zero-sum game duality would extend to deterministic committees.

All standard candidate-approval rules either already fail weaker representation axioms such as EJR or fail core stability. In particular, Aziz *et al.* (2017) have shown that PAV satisfies EJR, but may produce non-core-stable candidate committees even for candidate-approval elections for which core-stable candidate committees are known to exist.

By contrast, we show that PAV guarantees core stability in the party-approval setting. We follow the structure of the aforementioned proof showing that PAV satisfies EJR for candidate-approval elections (Aziz *et al.* 2017).

Theorem 2. *For every party-approval election, PAV chooses a core-stable committee.*

Proof. Consider a party-approval election (N, P, A, k) and let $W : P \to \mathbb{N}$ be the committee selected by PAV. Assume for contradiction that W is not core stable. Then, there is a nonempty coalition S and a committee $T : P \to \mathbb{N}$ such that $|T| \leqslant k\,|S|/n$ and $\sum_{p \in A_i} T(p) > \sum_{p \in A_i} W(p)$ for every voter $i \in S$.

Let $u_i(W)$ denote the number of seats in W that are allocated to parties approved by voter i, i.e., $u_i(W) = \sum_{p \in A_i} W(p)$. Furthermore, for a party p with $W(p) > 0$, we let $MC(p, W)$ denote the *marginal contribution* to the PAV score of allocating a seat to p, i.e., $MC(p, W) = \text{PAV}(W) - \text{PAV}(W - \{p\})$. Observe that $MC(p, W) = \sum_{i \in N_p} 1/u_i(W)$, where $N_p = \{i \in N \mid p \in A_i\}$. The sum of all marginal contributions satisfies

$$\sum_{p \in P} W(p)\, MC(p, W) = \sum_{p \in P} \sum_{i \in N_p} \frac{W(p)}{u_i(W)}$$
$$= \sum_{i \in N} \sum_{p \in A_i} \frac{W(p)}{u_i(W)} = |\{i \in N \mid u_i(W) > 0\}| \leqslant n.$$

Note that terms $MC(p, W)$ for $W(p) = 0$ and quotients $1/u_i(W)$ for $u_i(W) = 0$ are undefined in the calculation above, but that they only appear with factor 0.

It follows that the average marginal contribution of all k seats in W is at most n/k, and consequently, that there has to be a party p_1 with a seat in W such that $MC(p_1, W) \leqslant n/k$. Using a similar argument, we show that there is also a party p_2 with $T(p_2) > 0$ which would increase the PAV score by at least n/k if it received an additional seat in W:

$$\sum_{p \in P} T(p)\, MC(p, W + \{p\}) = \sum_{i \in N} \sum_{p \in A_i} \frac{T(p)}{u_i(W + \{p\})}$$
$$\geqslant \sum_{i \in S} \sum_{p \in A_i} \frac{T(p)}{u_i(W + \{p\})} = \sum_{i \in S} \sum_{p \in A_i} \frac{T(p)}{u_i(W) + 1}$$
$$\geqslant \sum_{i \in S} \sum_{p \in A_i} \frac{T(p)}{u_i(T)} = |\{i \in S \mid u_i(T) > 0\}| = |S|.$$

The second inequality holds because every voter in S *strictly* increases their utility when deviating from W to T; the last equality holds because every voter in S must get some representation in T to deviate. As desired, it follows that there has to be a party p_2 in the support of T with $MC(p_2, W + \{p_2\}) \geqslant |S|/|T| \geqslant n/k$.

If any of these inequalities would be strict, that is, if $MC(p_1, W) < n/k$ or $MC(p_2, W + \{p_2\}) > n/k$, then the committee $W - \{p_1\} + \{p_2\}$ would have a PAV-score of

$$\begin{aligned} &\mathrm{PAV}(W) - MC(p_1, W) + MC(p_2, W - \{p_1\} + \{p_2\}) \\ &\geqslant \mathrm{PAV}(W) - MC(p_1, W) + MC(p_2, W + \{p_2\}) \quad (1) \\ &> \mathrm{PAV}(W), \end{aligned}$$

which would contradict the choice of W.

Else, suppose that $MC(p, W) = n/k$ for all parties p in the support of W and $MC(p, W + \{p\}) = n/k$ for all parties p in the support of T. If there is a party p_1 in W that is approved by some voter $i \in S$, we can choose an arbitrary party p_2 from the support of T that i approves as well. Then, for voter i, the marginal contribution of p_2 in $W - \{p_1\} + \{p_2\}$ is $\frac{1}{u_i(W - \{p_1\} + \{p_2\})} = \frac{1}{u_i(W)}$, but the marginal contribution of p_2 in $W + \{p_2\}$ for i is only $\frac{1}{u_i(W + \{p_2\})} = \frac{1}{u_i(W) + 1}$. This implies $MC(p_2, W - \{p_1\} + \{p_2\}) > MC(p_2, W + \{p_2\})$, which makes inequality (1) strict and again contradicts the optimality of W.

Thus, one has to assume that no voter in S approves any party in the support of W. Pick an arbitrary p in the support of T, and recall that $MC(p', W + \{p'\}) = n/k$ for all p' in the support of T. Thus, all inequalities in the derivation of $\sum_{p \in P} T(p)\, MC(p, W + \{p\}) \geqslant |S|$ above must be equalities, which implies that this increase in PAV score must solely come from voters in S. Thus, there are at least n/k voters in S who are not represented at all in W, but commonly approve p. This would be a violation of EJR, contradicting the fact that PAV satisfies this axiom. □

Corollary 3. *The core of a party-approval election is nonempty.*

An immediate follow-up question is whether core-stable committees can be computed efficiently. PAV committees are known to be NP-hard to compute in the candidate-approval setting, and we confirm in Appendix B.1 that hardness still holds in the party-approval subdomain.

Equally confronted with the computational complexity of PAV, Aziz *et al.* (2018) proposed a local-search variant of PAV, which runs in polynomial time and guarantees EJR in the candidate-approval setting. Using the same approach, we can find a core-stable committee in the party-approval setting. We defer the proof to Appendix B.2.

Theorem 4. *Given a party-approval election, a core-stable committee can be computed in polynomial time.*

Theorem 2 motivates the question of whether other candidate-approval rules satisfy stronger representation axioms when restricted to the party-approval subdomain. We have studied this question for various rules besides PAV, and the answer was always negative.[6]

While the party-approval setting does not reduce the complexity of computing PAV, it allows us to efficiently check whether a given committee provides EJR or PJR; both problems are coNP-hard in the candidate-approval setting (Aziz *et al.* 2017; Aziz *et al.* 2018). For EJR, this follows from coherence becoming simpler for party-approval elections. Our algorithm for checking PJR employs submodular minimization. For details, we refer to Appendix B.3.

5 Discussion

In this paper, we have initiated the axiomatic analysis of approval-based apportionment. On a technical level, it would be interesting to see whether the party-approval domain allows us to satisfy other combinations of axioms that are not known to be attainable in candidate-approval elections. For instance, the compatibility between strong representation axioms and certain notions of support monotonicity is an open problem (Sánchez-Fernández and Fisteus 2019).

We have presented our setting guided by the application of apportioning parliamentary seats to political parties. We believe that this is an attractive application worthy of practical experimentation. Our formal setting has other interesting applications. An example would be participatory budgeting settings in which the provision of items of equal cost is decided, where the items come in different types. For instance, a university department could decide how to allocate Ph.D. scholarships across different research projects, in a way that respects the preferences of funding organizations.

As another example, the literature on multiwinner elections suggests many applications to recommendation problems (Skowron *et al.* 2016). For instance, one might want to display a limited number of news articles, movies, or advertisements in a way that fairly represents the preferences of the audience. These preferences might be expressed not over individual pieces of content, but over content producers (such as newspapers, studios, or advertising companies), in which case our setting provides rules that decide how many items should be contributed by each source. Expressing preferences on the level of content producers is natural in

[6]We consider the candidate-approval rules SeqPAV, RevSeqPAV, Approval Voting (AV), SatisfactionAV, MinimaxAV, SeqPhragmén, MaxPhragmén, VarPhragmén, Phragmén-STV, MonroeAV, Greedy-MonroeAV, GreedyAV, HareAV, and Chamberlin–CourantAV. Besides EJR and core stability, we consider JR and PJR (see Footnote 2). Definitions and results can be found in Appendix A.

repeated settings, where the relevant pieces of content change too frequently to elicit voter preferences on each occasion. Besides, content producers might reserve the right to choose which of their content should be displayed.

In the general candidate-approval setting, the search continues for rules that satisfy EJR and committee monotonicity, or core stability. But for the applications mentioned above, these guarantees are already achievable today.

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Appendix

The appendix is organized as follows.

In **Appendix A** we investigate for several candidate-approval rules whether they fulfill stronger axiomatic properties in the party-approval subdomain. To this end, we introduce all studied axioms and rules in Appendices A.1 and A.2, respectively, and present the results in Appendix A.3. Table 1 summarizes the results of this section.

In **Appendix B** we analyze the introduced axioms and rules from a computational complexity perspective. First, we show that computing a committee with maximum PAV score (which is core stable due to Theorem 2) as well as computing a winning committee for MaxPhragmén is still NP-hard in the party-approval subdomain (Appendix B.1). Second, we show that a core-stable committee can be computed in polynomial time (Appendix B.2). Third, we provide polynomial-time algorithms for checking whether a given committee satisfies EJR or PJR for a party-approval election (Appendix B.3).

A Axiomatic Study

A.1 Weaker Representation Axioms

Let (N, P, A, k) be a party-approval election and $W : P \to \mathbb{N}$ be a committee.

The committee W provides *justified representation* (JR) if there is no coalition of voters $S \subseteq N$ of size at least n/k, all of whose voters commonly approve of a party, but $\sum_{p\in A_i} W(p) = 0$ for all $i \in S$. A voting rule satisfies JR if it returns committees providing JR for every party-approval election.

The committee W provides *proportional justified representation (PJR)*, if there is no coalition of voters $S \subseteq N$, all of whose voters commonly approve a party, but $\sum_{p\in\bigcup_{i\in S} A_i} W(p) < q(S)$. A voting rule satisfies PJR if it returns committees providing PJR for every party-approval election.

A.2 Candidate-Approval Rules

In the following we introduce approval-based committee voting rules that were repeatedly studied in the literature. Note that we use the language of the candidate-approval setting and in particular, W is a set (not a multiset) of candidates. In order to apply the described rules in our setting, we can transform any party-approval election to a candidate-approval election by introducing k clones of each party. For a formal description of the transformation see Section 4.2 in the main part of the paper.

Sequential PAV (SeqPAV) and Reverse Sequential PAV (RevSeqPAV) Additionally to the PAV rule, Thiele proposed two strongly related voting rules which run in polynomial time and greedily try to maximize the PAV score: SeqPAV and RevSeqPAV, both of which construct committees in a sequential manner (Thiele 1895). SeqPAV starts with the empty committee and iteratively adds the candidate that increases the PAV score the most, until k candidates are selected. RevSeqPAV initially assumes every candidate to be winning; afterwards, the candidate whose removal from the committee decreases the committee's PAV score the least is removed, until there are only k candidates left.

Approval Voting (AV) Approval voting is the straightforward generalization of Approval Voting for single-winner elections (Brams and Fishburn 2007). The committee consists of the k candidates with the largest number of voters who approve the respective candidate. Thus, approval voting returns the set $W \subseteq C$ of size $|W| = k$ that maximizes the approval score $\sum_{c\in W} |N_c|$.

Satisfaction Approval Voting (SAV) SAV selects the k candidates with the highest *satisfaction score* $s(c) = \sum_{i\in N_c} \frac{1}{|A_i|}$ (Brams and Kilgour 2014). Thereby, SAV outputs the committee $W \subseteq C$ with $|W| = k$ that maximizes the score

$$\sum_{i\in N} \frac{|A_i \cap W|}{|A_i|}.$$

Minimax Approval Voting (MAV) MAV returns the committee that minimizes the maximum Hamming distance of the committee to any approval ballot (Brams *et al.* 2007). The Hamming distance between a ballot and a committee is simply the symmetric difference between the two candidate sets, i.e. $d(X, Y) = |X \setminus Y| + |Y \setminus X|$. MAV outputs the committee W with minimal score $\max\{d(W, A_i) : i \in N\}$.

Phragmén's rules (MaxPhragmén, VarPhragmén, Seq-Phragmén, and Phragmén-STV) Phragmén's voting rules, which were like PAV developed in the late 19th century.[7] The central concept behind his rules is the idea that every candidate in the committee carries some "load", which has to be distributed among the voters who approve this candidate. Phragmén's rules then aim to select a committee where this load can be distributed evenly among the voters.

Following Brill *et al.* (2017) for the formal definition, we call a real-valued vector $(x_{i,c})_{i\in N,c\in C}$ a *load distribution* if the following properties hold:

$$0 \leqslant x_{i,c} \leqslant 1 \qquad \text{for } i \in N, c \in C, \tag{2}$$

$$x_{i,c} = 0 \qquad \text{if } c \notin A_i, \tag{3}$$

$$\sum_{i\in N}\sum_{c\in C} x_{i,c} = k, \tag{4}$$

$$\sum_{i\in N} x_{i,c} \in \{0, 1\} \qquad \text{for } c \in C. \tag{5}$$

In this definition, $x_{i,c}$ represents the load of candidate c on voter i. Thus, the load that voter i carries is $\sum_{c\in C} x_{i,c}$. Properties (4) and (5) ensure that each load distribution corresponds to a committee of size k—namely, every candidate whose total load is one is part of the committee, while candidates whose total load is zero are not.

[7]Phragmén's original papers are written in French and Swedish (Phragmén 1894; Phragmén 1895; Phragmén 1896; Phragmén 1899); an English account of this work was composed by Janson (2016).

There are different ways of measuring how balanced a load distribution is. This corresponds to different Phragmén rules. The first rule **MaxPhragmén** minimizes the maximal total load of any voter. Formally, MaxPhragmén returns the committee that corresponds to the load distribution $(x_{i,c})$ where $\max_{i\in N}\sum_{c\in A_i} x_{i,c}$ is minimal.[8] The second rule **VarPhragmén** minimizes the variance of the voter loads by computing the load distribution that minimizes $\sum_{i\in N}(\sum_{c\in A_i} x_{i,c})^2$ and returning the corresponding committee. The third rule **SeqPhragmén** constructs it's committee sequentially by starting with an empty committee and then iteratively adding the candidate that increases the maximum voter load the least. See Algorithm 1 for a formal definition.

Algorithm 1 SeqPhragmén (Candidate-Approval Rule)

```
function SEQPHRAGMÉN(N, C, A, k)
    W ← ∅
    x_i ← 0 for every i ∈ N
    while |W| < k do
        s_c ← (1 + ∑_{i∈N_c} x_i)/|N_c| for every c ∈ C \ W
        c* ← argmin_{c∈C\W} s_c
        W ← W ∪ {c*}
        x_i ← s_{c*} for every i ∈ N_{c*}
    end while
    return W
end function
```

If we want to make use of **MaxPhragmén** or **VarPhragmén** in the party-approval setting, we can do so without utilizing the previously described embedding of party-approval elections into the space of candidate-approval elections. We do so by simply treating the parties as candidates and slightly adjusting the definition of a load distribution such that constraint (2) becomes $0 \leqslant x_{i,p} \leqslant k$ and constraint (5) becomes $\sum_{i\in N} x_{i,c} \in [k]\ \forall\, p \in P$.

Additionally, Phragmén developed a voting rule which is closely related to the well-known Single Transferable Vote system (Janson 2016; Sánchez-Fernández *et al.* 2018; Camps *et al.* 2019). We refer to this rule as **Phragmén-STV**; Camps *et al.* (2019) call it the Eneström-Phragmén method. In contrast to the previous three algorithms, it does not make use of load distributions. The core idea of the sequential procedure is the following: Initially, every voter has weight 1 and every candidate's score is the sum of all approving voters' weights. In every round, the candidate with the highest score s is added to the committee. If a voter i with weight f_i approves the candidate who is added to the committee, then their weight will be updated to $f_i \cdot (s - n/k)/s$ if $s > n/k$, and to 0 otherwise. This process is repeated until all k seats are assigned.

Monroe's rule (MonroeAV and GreedyMonroeAV) MonroeAV was originally intended for linear preferences of voters (Monroe 1995). Every voter is represented by exactly one winning candidate, but with the restriction that every candidate can only represent $\lfloor n/k\rfloor$ or $\lceil n/k\rceil$ many voters. Formally, a mapping $\pi : N \to W$ for a given committee $W \subseteq C$ describes that voter $i \in N$ is represented by $\pi(i)$. Such a mapping is a valid mapping if $|\pi^{-1}(c)| \in \{\lfloor n/k\rfloor, \lceil n/k\rceil\}$ for all $c \in W$. The score of a valid mapping π is the number of voters who are represented in π by a candidate they approve. The Monroe score of a committee $W \subseteq C$ is the maximum score of any valid mapping for W, and MonroeAV returns the committee with the highest Monroe score.

Skowron *et al.* (2015) proposed a greedy variant of MonroeAV, which was later adopted for the approval setting by Sánchez-Fernández *et al.* (2017). We refer to this algorithm as GreedyMonroeAV. GreedyMonroeAV constructs its committee in k rounds. It is initialized with $N' = N$ and $W = \emptyset$. In round $t \in \{1, \dots, k\}$, the algorithm selects $c_t \in C \setminus W$ and $N_t \subseteq N'$ such that the number of voters in N_t who approve c_t is maximal. Meanwhile, N_t is restricted to be of size $\lceil n/k\rceil$ if $t \leqslant n - k\lfloor n/k\rfloor$ and to be of size $\lfloor n/k\rfloor$ otherwise. At the end of each round, c_t is added to W and the voters in N_t are removed from N'.

GreedyAV and HareAV In GreedyAV (Aziz *et al.* 2017), the committee is constructed iteratively by adding the candidate that satisfies the most voters, who do not approve any of the previously added candidates. More formally, the algorithm initially starts with $A' = A$ and $W = \emptyset$. While $|W| < k$ and A' is nonempty, the candidate $c \in C \setminus W$ with highest approval score[9] with regard to A' is chosen and added to W. Then, the ballots of all voters in N_c are removed from A' and the next candidate is picked analogously. If at some point $|W| < k$ and A' is empty, then W is filled up with arbitrary candidates until $|W| = k$.

HareAV is a variant of GreedyAV where—after $c \in C$ is added to the committee—not the ballots of all voters in N_c are removed from A', but instead only $\min\{\lceil n/k\rceil, |N_c|\}$ many ballots are removed.

Chamberlin-Courant's rule (CCAV) Chamberlin and Courant (1983) developed a multiwinner voting rule which was originally defined for voters with linear preferences. We define an adapted version for the setting of approval votes. The utility of a voter i for a committee W is defined to be 1 if $W \cap A_i \neq \emptyset$ and 0 otherwise. CCAV selects a committee that maximizes the sum of voters' utilities.

A.3 Results

We summarize the axiomatic properties of all above introduced voting rules when applied in the party-approval subdomain in Table 1. In the following we present the corresponding proofs following the table from top to bottom.

The proof that PAV satisfies core stability can be found in Section 4.3 of the main part of the paper.

We start by showing that, equivalently to the candidate-approval setting, the two greedy variants of PAV, SeqPAV and RevSeqPAV do not fulfill JR in the party-approval setting.

[8] For technical reasons, the usual lexicographic tie-breaking does not suffice here and has to be replaced with a more involved tie-breaking mechanism; for details, see the work of Brill *et al.* (2017).

[9] As usual, ties are broken lexicographically.

Rule	JR	PJR	EJR	Core Stability
PAV	✓	✓	✓	✓
SeqPAV	-	-	-	-
RevSeqPAV	-	-	-	-
AV	-	-	-	-
SAV	-	-	-	-
MAV	-	-	-	-
SeqPhragmén	✓	✓	-	-
MaxPhragmén	✓	✓	-	-
VarPhragmén	✓	-	-	-
Phragmén-STV	✓	✓	-	-
MonroeAV	✓	-	-	-
GreedyMonroeAV	✓	-	-	-
GreedyAV	✓	-	-	-
HareAV	✓	-	-	-
CCAV	✓	-	-	-

Table 1: The table contains a summary of the axiomatic properties of candidate-approval rules within the subdomain of party-approval elections.

Indeed, the counterexamples presented by Aziz *et al.* (2017) and Aziz (2017) also show that both rules violate JR for party-elections.

Proposition 5. *In party-approval elections, SeqPAV violates JR for $k \geqslant 10$ and RevSeqPAV violates JR for $k \geqslant 5$.*

Proof. The adapted counterexample by Aziz *et al.* (2017) has $n = 1199$ voters and parties $P = \{P_1, \ldots, P_{11}\}$. First, assume $k = 10$. The ballot profile is as follows:

$$\begin{array}{lll} 81 \times \{P_1, P_2\}, & 81 \times \{P_1, P_3\}, & 80 \times \{P_2\}, \\ 80 \times \{P_3\}, & 81 \times \{P_4, P_5\}, & 81 \times \{P_4, P_6\}, \\ 80 \times \{P_5\}, & 80 \times \{P_6\}, & 49 \times \{P_7, P_8\}, \\ 49 \times \{P_7, P_9\}, & 49 \times \{P_7, P_{10}\}, & 96 \times \{P_8\}, \\ 96 \times \{P_9\}, & 96 \times \{P_{10}\}, & 120 \times \{P_{11}\}. \end{array}$$

In the beginning of the SeqPAV computation, both P_1 and P_4 have a score of $81 + 81 = 162$. As they are approved by disjoint set of voters, in the first two iterations both parties receive each one seat. In the third iteration, P_7 has the highest score of $3 \cdot 49 = 147$ and receives a seat. In the remaining iterations, P_2, P_3, P_5, P_6 have a score of $80 + \frac{81}{2} = 120.5$, and P_8, P_9, P_{10} have a score of $96 + \frac{49}{2} = 120.5$, while P_{11} has only 120 score points. The parties already present in the committee all have a score smaller than 120. Furthermore, giving each of the parties with score 120.5 one seat does not affect the score of the other parties—therefore all of them receive one seat. Thus, $P_1, \ldots, P_{10}$ receive each one seat in the committee, making the $120 > n/k$ voters approving only P_{11} unrepresented. This violates JR.

For $k \geqslant 11$, one can add $k - 10$ additional parties $P_{12}, \ldots, P_{k+1}$ and $120 \cdot (k-10)$ additional voters to the ballot profile, such that for each new party $P_j, 12 \leqslant j \leqslant k+1$, 120 new voters approve only P_j. This does not affect the SeqPAV computation described above. After the first ten iterations, the remaining $k - 10$ seats are given to the parties $P_{11}, \ldots, P_{k+1}$. Because there are more parties than seats left, one of these parties does not receive a seat, resulting again in 120 cohesive voters who are unrepresented. This violates JR, as $120 > 120 - \frac{1}{k} = \frac{1199+120\cdot(k-10)}{k} = n/k$.

In an analogous way, we can adapt the counterexample by Aziz (2017), showing that RevSeqPAV violates JR for $k \geqslant 5$. □

When applied in the party-approval setting, the voting rules AV and SAV select the party with the highest approval respectively satisfaction score and allocate *all* available seats to this party. Evidently, both rules do not satisfy JR. The same holds for MAV.

Proposition 6. *AV, SAV, and MAV violate JR in the party-approval setting for $k \geqslant 3$.*

Proof. For AV and SAV, consider a party-approval election with two parties A and B, and $n = 3k$ voters with approval ballots $A_1 = \ldots = A_{2k} = \{A\}, A_{2k+1} = \ldots = A_{3k} = \{B\}$. In AV's and SAV's committee, A receives all seats, resulting in $k \geqslant 3 = n/k$ uninamous unrepresented voters. This violates JR.

For MAV, consider an election with four parties A, B, C, D and $n = 2k$ voters with ballots $A_1 = \ldots = A_k = \{A, B, C\}, A_{k+1} = \ldots = A_{2k} = \{D\}$. JR would require D to receive at least one seat. However, MAV does not select such a committee: First, consider the committee W_A where A receives all seats. In the induced candidate-approval committee, the Hamming distance between any approval set A_i and the committee W_A is $d(W_A, A_i) = 2k$.[10] Secondly, consider any committee W_D where D has at least one seat. Then, we have $d(W_D, A_1) \geqslant 2k+2$ in the induced candidate-approval election. Thus, W_A has a smaller maximal distance than any W_D. As a result, MAV does not return a committee that allocates at least one seat to D, a contradiction to JR. □

Proposition 7. *SeqPhragmén fails EJR for $k \geqslant 282$ on party-approval elections.*

Proof. Fix a natural number $k \geqslant 282$. We construct a party-approval election with parties A, B, C, D, E, X. The ballot profile of the $n = 2k$ many voters is as follows:

$$1 \times \{A, X\},\ 1 \times \{B, X\},\ 1 \times \{C, X\},\ 1 \times \{D, X\},$$
$$7 \times \{A, B, C, D\},\ (2k - 11) \times \{E\}.$$

We first ignore the voters approving E and focus on the 11 remaining ones. Initially, adding a seat to A, B, C, or D would increase the maximal voter load to $\frac{1}{8}$, while giving X one seat would increase it to $\frac{1}{4}$. Without loss of generality, assume A receives this seat. Then, giving the next seat to B, C, or D would increase the maximal load to $(\frac{7}{8} + 1) \cdot \frac{1}{8} = \frac{15}{64}$; giving it to A would increase it to $\frac{1+1}{8} = \frac{16}{64}$, and giving the seat to X would increase it to $\frac{\frac{1}{8}+1}{4} = \frac{18}{64}$. Thus, we can assume B receives the seat. Analogously, the next two seats are allocated to C and D, respectively—the exact computations can be found in Table 2. The fifth seat would

[10] Keep in mind that in the induced candidate-approval election, the approval sets contain k copies of every party as candidates.

then be allocated again to A, increasing the maximal voter load to $\frac{16473}{32768} \approx 0.50272$.

Taking the voters for E into account would not affect the computation above, because all voters who approve E do not approve any other party. Thus, in every SeqPhragmén iteration, either E receives a seat or one of A, B, C, D receives a seat, until A, B, C, D all have one seat. Adding a seat to E increases the load of a voter approving E by $\frac{1}{2k-11}$. Thus, if $k-4$ seats are allocated to E, each E-voter would have a load of $\frac{k-4}{2k-11}$. Observe that $\lim_{k\to\infty} \frac{k-4}{2k-11} = \frac{1}{2}$ and indeed, $\frac{k-4}{2k-11} < 0.50272$ for all $k \geqslant 282$. Therefore, SeqPhragmén returns a committee where A, B, C, D each receive one seat and E receives the remaining $k-4$ seats.

This is a contradiction to EJR: Since $n/k = 2$, EJR demands one of the four voters approving X to be represented at least twice in the committee. This is not the case in the committee selected by SeqPhragmén. □

In order to prove that MaxPhragmén and VarPhragmén violate EJR, we take a detour of arguing that another proportional representation axiom (introduced by Sánchez-Fernández *et al.* 2017) which is incompatible to EJR in the candidate-approval setting remains incompatible to EJR in the party-approval setting. Since MaxPhragmén and VarPhragmén fulfill this property, this suffices to prove our claim.

The axiom reflects the idea that every candidate represents n/k voters who are all approve of this representation. More precisely, given an election (N, C, A, k) where k divides n, a committee W satisfies *perfect representation (PR)* if it is possible to partition the voters into distinct sets $N_1, \ldots, N_k$ all of size n/k and assign to each $N_q, q \in \{1, \ldots, k\}$, a different winning candidate $c_q \in W$ such that all voters in N_q approve c_q. A voting rule satisfies PR if it always returns a committee that satisfies PR, whenever such a committee exists.

The proof by Sánchez-Fernández *et al.* (2017) showing that EJR and PR are incompatible for candidate-approval elections can be adapted straightforwardly to the party-approval subdomain by interpreting candidates as parties.

Proposition 8. *There exists no party-approval voting rule which satisfies EJR and PR.*

Phragmén's optimization rules, MaxPhragmén and VarPhragmén, both satisfy PR in the candidate-approval setting (Brill *et al.* 2017) and hence also in the party-approval setting.

Corollary 9. *MaxPhragmén and VarPhragmén do not satisfy EJR for party-approval elections.*

In the following we show that also Phragmén-STV does not show improved axiomatic properties for party-approval elections.

Proposition 10. *Phragmén-STV fails EJR for $k \geqslant 18$ on party-approval elections.*

Proof. For $k \geqslant 18$, consider an election with $n = 120k$ voters and $k+1$ parties $A, X_1, \ldots, X_k$. The ballot profile is as follows:

$$
\begin{aligned}
&120 \times \{A, X_1\}, \\
&120 \times \{A, X_2\}, \\
&122 \times \{X_1, X_3\}, \\
&70 \times \{X_2, X_4\}, \\
&120 \times \{X_4, X_5\}, \\
&121 \times \{X_5, X_6\}, \\
&61 \times \{X_3\}, \\
&50 \times \{X_4\}, \\
&65 \times \{X_6\}, \\
&109 \times \{X_j\} \qquad\qquad \text{for } j \in \{7, \ldots, 15\}, \\
&110 \times \{X_j\} \qquad\qquad \text{for } j \in \{16, 17, 18\}.
\end{aligned}
$$

If $k > 18$, we also add 120 voters approving $\{X_j\}$ for every $j \in \{19, \ldots, k\}$.

At first, let us only consider the parties $A, X_1, \ldots, X_6$. In Table 3, the Phragmén-STV calculation for an election restricted to these parties is described. Note that in the first 6 iterations, the parties $X_1, \ldots, X_6$ each receive one seat and all have, when selected as winners, a score that exceeds 120. Afterwards, every party has a score strictly smaller than 109.

Furthermore, observe that the parties $X_7, \ldots, X_k$ are all approved by voters who only approve this one particular party. As a result, their scores are not affected when other parties receive a seat. The parties $X_7, \ldots, X_{15}$ have a score of 109, X_{16}, X_{17}, X_{18} a score of 110, and $X_{19}, \ldots, X_k$ (if existing) a score of 120. When any of these parties receive a seat, their score is decreased to 0, as they are all approved by at most n/k voters.

Together, this shows that Phragmén-STV firstly allots each one seat to $X_1, \ldots, X_6$. Then, the score of the parties $A, X_1, \ldots, X_6$ is always smaller than 109, and therefore, $X_7, \ldots, X_k$ all receive a seat, which fills the committee. Thus, in the committee selected by Phragmén-STV, $X_1, \ldots, X_k$ each receive one seat. However, the $240 = 2n/k$ voters who approve A form a cohesive group, where at least one voter should be represented by at least two seats according to EJR. This is not the case in the committee generated by Phragmén-STV. □

Proposition 11. *GreedyAV and CCAV do not satisfy PJR for $k \geqslant 3$ on party-approval elections.*

Proof. Regarding GreedyAV with $k \geqslant 3$, consider an election with $n = 3k$ many voters and a parties $A, X_1, \ldots, X_k$. The ballots are $A_i = \{X_i\}$ for $1 \leqslant i \leqslant k$ and $A_i = \{A\}$ for $k < i \leqslant 3k$. GreedyAV first gives one seat to party A and afterwards $k-1$ many seats, each to a different party from $\{X_1, \ldots, X_k\}$. Then, $N^* = \{k+1, \ldots, n\}$ is a set of $2k$ many voters with equal ballot who are only represented by one approved candidate. However, for $\ell = 2$ we have $2k \geqslant 6 = \ell n/k$ and therefore PJR claims that the voters in N^* must be represented by at least two seats.

It is not hard to see that, for the above constructed election, CCAV selects the same committee like GreedyAV. To see this note that the CCAV-utility of this committee is $3k-1$.

Party	Iteration 1	Iteration 2	Iteration 3	Iteration 4	Iteration 5
A	**0.125**	0.25	0.34570	0.42944	**0.50272**
B	0.125	**0.23438**	0.35938	0.44312	0.51639
C	0.125	0.23438	**0.33008**	0.45508	0.52835
D	0.125	0.23438	0.33008	**0.41382**	0.53882
X	0.25	0.28125	0.33984	0.42236	0.52582

Table 2: The SeqPhragmén computation for the profile in Proposition 7, when the party E is ignored. The table shows the values of s_p for a party p in the first five iterations, rounded to five significant digits. The bold entries denote which party receives a seat (with lexicographic tie-breaking).

Party	Round 1	Round 2	Round 3	Round 4	Round 5	Round 6	Round 7
A	240.0	179.86	179.86	103.26	103.26	103.26	**103.26**
X_1	**242.0**	120.71	120.71	120.71	120.71	120.71	60.14
X_2	190.0	190.0	**190.0**	68.71	45.96	45.96	45.96
X_3	183.0	121.86	121.86	121.86	121.86	**121.86**	0.57
X_4	240.0	240.0	179.61	**134.92**	13.64	13.64	13.64
X_5	241.0	**241.0**	119.71	119.71	66.13	7.86	7.86
X_6	186.0	186.0	125.11	125.11	**125.11**	3.82	3.82

Table 3: The Phragmén-STV computation for the restricted profile in Proposition 10. The table shows the scores of the parties in the first seven iterations. The bold entries denote the party with the highest score.

Now, consider a committee which gives two seats to one of the parties. Such a committee has a CCAV-utility of at most $3k - 2$. Hence, both GreedyAV and CCAV fail PJR. □

Proposition 12. *MonroeAV, GreedyMonroeAV and HareAV violate PJR for $k \geqslant 6$ on party-approval elections.*

Proof. For $k \geqslant 6$, consider an election with $n = k + 3$ voters and parties $A, X_1, \ldots, X_{k-3}$. In the election, 6 voters approve only A; furthermore for $j \in \{1, \ldots, k-3\}$, there is one voter who approves only X_j. For this election, MonroeAV, GreedyMonroeAV and HareAV all return a committee where A receives 3 seats, and the remaining seats are distributed among $X_1, \ldots, X_{k-3}$ (none of them receives more than one seat).

Since for $k \geqslant 6$ there are $6 \geqslant 4 + \frac{12}{k} = 4n/k$ many voters approving only A, PJR would demand 4 seats for A. This is not satisfied by the computed committee. □

B Computational Aspects

B.1 Hardness of PAV and MaxPhragmén in the Party-Approval Subdomain

We show NP-hardness by reduction from the classical NP-complete problem Independent Set (Garey and Johnson 1979).

Independent Set
Input: Undirected graph $G = (V, E)$, $t \in \mathbb{N}$.
Question: Is there a vertex subset $V' \subseteq V$ of size $|V'| = t$ such that no two vertices in V' are connected by an edge in G?

This problem is NP-hard even when restricted to cubic graphs (where every vertex has degree 3). With this definition in mind, the two hardness results can be formulated. Our reduction is a simplified version of the reduction proposed by Aziz et al. (2015, Theorem 1).

Theorem 13. *For a given threshold s, deciding whether there exists a committee with PAV score at least s is NP-hard in the party-approval subdomain.*

Proof. For a given graph $G = (V, E)$ and independent set size $t \in \mathbb{N}$, we construct a party-approval election (N, P, A, k) is constructed. For each vertex $v \in V$, there is a fresh party $p_v \in P$. For every edge $e = \{u, v\} \in E$, there is one voter in N who approves exactly p_u and p_v. Let $deg(v)$ be the degree of v, and let $deg(G)$ denote the maximal vertex degree of G. Then, for every $v \in V$, insert $deg(G) - deg(v)$ many additional dummy voters who each approve only p_v. Lastly, we set $k = t$.

This reduction is obviously polynomial in the size of G. We show that G has an independent set of size t iff there is a committee W for the election (N, P, A, k) with PAV$(W) \geqslant s = deg(G)\, t$.

"⇒": Assume that G has an independent set $V' \subseteq V$ of size $|V'| = t$. Consider the committee W where for every vertex $v \in V'$, the party p_v receives exactly one seat (thus, the committee has the proper size $k = t$). Each party p_v is approved by $deg(G) - deg(v)$ many dummy voters and by $deg(v)$ many edge voters. Because V' is an independent set, no voter approves more than one party in the committee, and thus only has a single seat on the committee belonging to an approved party. Consequently, the total PAV score of W is exactly $deg(G)\, t$.

"⇐": Assume that W is a committee with PAV$(W) \geqslant deg(G)\, t$ for the constructed election. Every party p_v is approved by exactly $deg(G)$ many voters and therefore, giving one seat to p_v in the committee can increase the PAV-score by at most $deg(G)$. As there are only t seats available, every seat assignment has to increase the PAV-score by exactly $deg(G)$. In order to achieve an increase of $deg(G)$ when adding a seat to p_v, all the voters who approve p_v must have been previously completely unrepresented. Thus, all parties present in W receive only one seat and do not have any common approving voters. By construction, this implies that the set of vertices $\{v \in V : W(p_v) > 0\}$ corresponding to W is an independent set of size t. □

Theorem 14. *Computing a winning committee for Max-Phragmén is NP-hard in the party-approval subdomain.*

Proof. We prove that the following problem is NP-hard:

PARTY APPROVAL MAXPHRAGMÉN
Input: party-approval election (N, P, A, k), distribution bound $s \in \mathbb{R}$.
Question: Is there a load distribution $(x_{i,p})$ such that $\max_{i \in N} \sum_{p \in A_i} x_{i,p} \leqslant s$?

Recall that we consider the adjusted definition of load distributions in the party-approval setting, which we introduced in Appendix A.2. Similarly to Theorem 13, we use a polynomial reduction from INDEPENDENT SET, this time taking advantage of the fact that INDEPENDENT SET is NP-hard even on cubic graphs. The reduction is a variant of the one by Brill *et al.* (2017), which shows that MaxPhragmén is NP-hard in the candidate-approval setting.

Given a cubic graph $G = (V, E)$ and independent set size $t \in \mathbb{N}$, we define the following party-approval election: For every vertex $v \in V$, there is a party $p_v \in P$. Additionally, for every edge $e = \{u, w\} \in E$, there is a voter in N who approves exactly p_u and p_w. The committee shall be as large as the independent set, that is, $k = t$. To prove that this reduction is sound, we show that G has an independent set of size t if and only if there is a load distribution $(x_{i,p})$ with $\max_{i \in N} \sum_{p \in A_i} x_{i,p} \leqslant \frac{1}{3}$.

"⇒": Assume G has an independent set $V' \subseteq V$ of size $|V'| = t$. Because G is cubic, every party in the created election is approved by exactly 3 voters. We define a valid load distribution, in which every party corresponding to a vertex in V' creates a load of $\frac{1}{3}$ on every approving voter. (This also implies that in the induced committee, the parties corresponding to V' receive exactly one seat.) Because V' is an independent set, no voter receives load from multiple parties, and hence the maximal total load of every voter is $\frac{1}{3}$.

"⇐": Assume there is a load distribution $(x_{i,p})$ such that $\max_{i \in N} \sum_{p \in A_i} x_{i,p} \leqslant \frac{1}{3}$. Since every party is approved by exactly 3 voters, it follows that $x_{i,p} = \frac{1}{3}$ for a voter i who approves a party p that receives a seat in the induced committee. Consequently, no party receives more than one seat in the induced committee and no voter approves more than one party in the committee. Thus, the committee induces the independent set $\{v \in V : x_{i,p_v} > 0 \text{ for some } i \in N\}$ of size t. □

B.2 Efficient Computation of Core-Stable Committees

In the following we prove that in the party-approval subdomain, core-stable committees can be computed in polynomial time. We make use of a local search procedure, introduced for the candidate-approval setting by Aziz *et al.* (2018), which approximates a local maximum of the PAV score function. Aziz *et al.* (2018) show that their algorithm runs in polynomial time and returns committees providing EJR. For party-approval elections, we show that, by a minor adjustment of the algorithm, committees computed by LS-PAV fulfill core stability. In Algorithm 2 we slightly adjust the original definition by parameterizing the procedure by the approximation threshold ϵ. Note that, once again, the algorithm is defined in terms of candidate-approval elections; in Section 4.2 we show how to apply candidate-approval rules to party-approval elections.

Algorithm 2 LS-PAV (Candidate-Approval Rule)

function LS-PAV(N, C, A, k, ϵ)
  $W \leftarrow k$ arbitrary candidates from C
  while $\exists\, c \in W, c \in C \setminus W$ such that $PAV(W \setminus \{c\} \cup \{c'\}) \geqslant PAV(W) + \epsilon$ **do**
    $W \leftarrow W \setminus \{c\} \cup \{c'\}$
  end while
  return W
end function

Theorem 4. *Given a party-approval election, a core-stable committee can be computed in polynomial time.*

Proof. In order to prove the theorem we show that LS-PAV with threshold $\epsilon = \frac{1}{(1+2(k-1))(k-1)k}$ always selects a committee W from the core when $k > 1$. Computing a core-stable committee for $k = 1$ is trivial. The proof is an adjustment of the proof of Theorem 2.

We define the utility of voter i w.r.t. committee W as $u_i(W) = \sum_{p \in A_i} W(p)$. For some party-approval election (N, P, A, k) let W be a committee selected by LS-PAV with ϵ and assume that W is not core stable. Hence, there exists $S \subseteq N, T : P \to \mathbb{N}, \ell \in [k]$ with $|S| \geqslant \ell\, n/k, |T| = \ell$ and

$$u_i(T) > u_i(W) \ \forall\, i \in S. \tag{6}$$

Following the proof of Theorem 2, we define the marginal contribution of a party p to the PAV score of W

$$\begin{aligned} MC(p, W) &= \text{PAV}(W) - \text{PAV}(W - \{p\}) \\ &= \sum_{i \in N : p \in A_i} \frac{1}{u_i(W)} \end{aligned}$$

and obtain an upper bound for the sum of the marginal contribution of all seats in W w.r.t. W, i.e.,

$$\sum_{p \in P} W(p) \cdot MC(p, W) \leqslant n \tag{7}$$

as well as a lower bound for the sum of marginal contribution of all seats in T to $W + \{p\}$, where p is the party corresponding to the seat,i.e.,

$$\sum_{p\in P} T(P)\cdot MC(p, W+\{p\}) \geqslant |S|. \tag{8}$$

Hence, there exists a party p_1 in the support of W for which

$$MC(p_1, W) \leqslant n/k \tag{9}$$

and there exists a party p_2 in the support of T for which

$$MC(p_2, W+\{p_2\}) \geqslant n/k \tag{10}$$

holds. We distinguish three cases:

Case 1: It holds that

1. $\exists\, p_1$ in supp. of $W : MC(p_1, W) \leqslant n/k - \epsilon$ **or**
2. $\exists\, p_2$ in supp. of $T : MC(p_2, W+\{p_2\}) \geqslant n/k + \epsilon$.

W.l.o.g. assume that the first condition is fulfilled. Then, we choose p_1 such that the condition is fulfilled and p_2 such that Eq. (10) holds. Observe that the set $W' = W - \{p_1\}+\{p_2\}$ yields a PAV score of more than $PAV(W)+\epsilon$, a contradiction to the termination of LS-PAV.

$$\begin{aligned}&\mathrm{PAV}(W')\\ &= \mathrm{PAV}(W) - MC(p_1, W) + MC(p_2, W-\{p_1\}+\{p_2\})\\ &\geqslant \mathrm{PAV}(W) - MC(p_1, W) + MC(p_2, W+\{p_2\})\\ &\geqslant \mathrm{PAV}(W) + \epsilon\end{aligned}$$

The last inequality follows from case condition 1 and Eq. (10). If the second condition holds, an analogous argument yields a contradiction.

Case 2: It holds that

1. $\forall\, p_1$ in supp. of $W : MC(p_1, W) > n/k - \epsilon$ **and**
2. $\forall\, p_2$ in supp. of $T : MC(p_2, W+\{p_2\}) < n/k+\epsilon$ **and**
3. $\exists\, i \in S : u_i(W) > 0$.

Applying Eq. (7) and case condition 1 we obtain an upper bound for $MC(p_1, W)$ for any p_1 in the support of W:

$$\begin{aligned}&MC(p_1, W)\\ &\leqslant n - \sum_{p\in P\setminus p_1} W(p)\; MC(p, W)\\ &\qquad -(W(p_1)-1)\; MC(p_1, W)\\ &< n-(k-1)\left(\frac{n}{k}-\epsilon\right) = \frac{n}{k}+(k-1)\epsilon.\end{aligned} \tag{11}$$

Analogously, applying Eq. (8) and case condition 2 we obtain an lower bound for $MC(p_2, W+\{p_2\})$ for any p_2 in the support of T:

$$\begin{aligned}&MC(p_2, W+\{p_2\})\\ &\geqslant |S| - \sum_{p\in P\setminus p_2} T(p)\; MC(p, W+\{p\})\\ &\qquad -(T(p_2)-1)\; MC(p_2, W+\{p_2\})\\ &> |S| - (l-1)\left(\frac{n}{k}+\epsilon\right) \geqslant \frac{n}{k}-(k-1)\epsilon.\end{aligned} \tag{12}$$

Subsequently, choose some $i \in S$ with $u_i(W) > 0$ (existence guaranteed by case condition 3) and a party p_1 from the support of W which is also included in the approval set of voter i, A_i. Then, choose a party p_2 in the support of T which is also approved by voter i but $W(p_2) < T(p_2)$ (existence guaranteed by the fact that voter i prefers committee T to committee W). Please note that the restrictions made by case conditions 1 and 2 already imply that p_1 and p_2 are different parties.[11]

For this choice of p_1 and p_2 we aim to show that the contribution of p_2 with respect to $W-\{p_1\}+\{p_2\}$ is strictly bounded away from the contribution of p_2 with respect to $W+\{p_2\}$. More precisely,

$$MC(p_2, W-\{p_1\}+\{p_2\}) \geqslant MC(p_2, W+\{p_2\}) + \frac{1}{k(k-1)}. \tag{13}$$

To this end recall that voter i supports party p_1 and hence

$$u_i(W-\{p_1\}) = u_i(W)-1. \tag{14}$$

Moreover, for all remaining voters $j \in N\setminus\{i\}$ it holds

$$u_j(W-\{p_1\}) \leqslant u_j(W). \tag{15}$$

Lastly, from i being in the deviator set S, we know that

$$u_i(W) \leqslant k-1. \tag{16}$$

For ease of presentation we define $N_p : \{i\in N : p \in A_i\}$ as the supporters of party p and $N_p^{-i} = N_p \setminus \{i\}$ as the supporters of party p without voter i. Putting it all together we get

$$\begin{aligned}&MC(p_2, W-\{p_1\}+\{p_2\})\\ &= \sum_{j\in N_{p_2}} \frac{1}{u_j(W-\{p_1\})+1}\\ &= \sum_{j\in N_{p_2}^{-i}} \frac{1}{u_j(W-\{p_1\})+1} + \frac{1}{u_i(W-\{p_1\})+1}\\ &\geqslant \sum_{j\in N_{p_2}^{-i}} \frac{1}{u_j(W)+1} + \frac{1}{u_i(W)}\\ &= \sum_{j\in N_{p_2}^{-i}} \frac{1}{u_j(W)+1} + \frac{1}{u_i(W)+1} + \frac{1}{(u_i(W)+1)u_i(W)}\\ &\geqslant MC(p_2, W+\{p_2\}) + \frac{1}{k(k-1)}.\end{aligned}$$

The first inequality holds due to Eq. (14) and Eq. (15) and the second due to Eq. (16).

Finally, making use of Eqs. (11) to (13), we can show that the committee $W' = W-\{p_1\}+\{p_2\}$ yields a PAV score

[11] Assume for contradiction that p_1 and p_2 are the same parties. Then, in particular it holds that $MC(p_1, W) = MC(p_2, W)$ and $MC(p_1, W+\{p_1\}) = MC(p_2, W+\{p_2\})$. Consider the difference $MC(p_1, W) - MC(p_1, W+\{p_1\})$. Note that we do not do any further assumptions in order to derive Eq. (13). Preempting Eq. (13), we know that $MC(p_1, W) - MC(p_1, W+\{p_1\}) \geqslant \frac{1}{k(k-1)}$ holds, but on the other hand we get from Eq. (11) and Eq. (12) that $MC(p_1, W) - MC(p_1, W+\{p_1\}) \leqslant 2(k-1)\epsilon$ holds, a contradiction.

that is greater than $\mathrm{PAV}(W)+\epsilon$, a contradiction.

$$\begin{aligned}
&\mathrm{PAV}(W') \\
&= \mathrm{PAV}(W) - MC(p_1,W) + MC(p_2, W-\{p_1\}+\{p_2\}) \\
&\geqslant \mathrm{PAV}(W) - MC(p_1,W) + MC(p_2, W+\{p_2\}) + \frac{1}{k(k-1)} \\
&> \mathrm{PAV}(W) - \frac{n}{k} - (k-1)\epsilon + \frac{n}{k} - (k-1)\epsilon + \frac{1}{k(k-1)} \\
&= \mathrm{PAV}(W) - 2(k-1)\epsilon + \frac{1}{k(k-1)} \\
&= \mathrm{PAV}(W) + \epsilon
\end{aligned}$$

The first inequality is due to Eq. (13) and the second due to Eqs. (11) and (12).

Case 3: Finally, suppose that we are neither in Case 1 nor in Case 2. It follows that $\sum_{i\in S} u_i(W) = 0$ but $\sum_{i\in S} u_i(T) \geqslant |S|$. Hence, there exists some p_2 in the support of T with at least $|S|/|T| \geqslant n/k$ supporters in S. This is a contradiction to the fact that LS-PAV fulfills EJR (and hence JR).[12]

It remains to show that LS-PAV for $\epsilon = \frac{1}{(1+2(k-1))(k-1)k}$ runs in polynomial time. We follow the proof by Aziz *et al.* (2018) showing that LS-PAV runs in polynomial time for $\epsilon' = n/k^2$. First note that the computations within every iteration of the algorithm run in polynomial time. It remains to show that the number of iterations is polynomial in n and k. The PAV score of a committee is upper bounded by $n \cdot H_k \in \mathcal{O}(n \ln k)$. The algorithm improves the PAV score of a committee in every iteration by at least ϵ. Hence, the running time is in $\mathcal{O}(nk^3 \ln k)$. ☐

B.3 Complexity of Checking EJR and PJR

Theorem 15. *Given a party-approval election (N, P, A, k) and a committee $W : P \to \mathbb{N}$, it can be checked in polynomial time whether W satisfies EJR.*

Proof. A committee W violates EJR iff there is a party $p \in P$ and an integer $1 \leqslant \ell \leqslant k$ such that there is a set S of voters supporting p of size at least $\ell\, n/k$ but such that $\sum_{p'\in A_i} W(p') < \ell$ for every voter $i \in S$. This condition can be checked in polynomial time.

For each such party p and integer ℓ, compute the number of voters supporting p who have a utility of less than ℓ. If this number is at least $\ell\, n/k$ at any point, a violation of EJR was found. Else, W provides EJR. A straightforward implementation of this algorithm has polynomial running time $\mathcal{O}(|P|\, k\, n)$. ☐

Checking PJR is slightly more involved, and uses techniques from submodular optimization. To recall, given a finite set U, a function $f : 2^U \to \mathbb{R}$ is *submodular* if for all subsets $X, Y \subseteq U$ with $X \subseteq Y$ and for every $x \in U \setminus Y$, it holds that

$$f(X \cup \{x\}) - f(X) \geqslant f(Y \cup \{x\}) - f(Y).$$

A submodular function $f : 2^U \to \mathbb{Z}$ can be minimized in time polynomial in $|U| + \log \max\{|f(S)| : S \subseteq U\}$ (Korte and Vygen 2018, Theorem 14.19). Applying this proposition, one can check whether a party-approval committee provides PJR in polynomial time.

Theorem 16. *Given a party-approval election (N, P, A, k) and a committee $W : P \to \mathbb{N}$, it can be checked in polynomial time whether W satisfies PJR.*

Proof. Fix $W : P \to \mathbb{N}$. At first, we define a function $s : 2^N \to \mathbb{N}$ such that

$$s(S) = \sum_{p \in \bigcup_{i\in S} A_i} W(p).$$

For any voter group $S \subseteq N$, $s(S)$ is the total number of seats that W allocates to parties approved by some voter in S.

W satisfies PJR iff there is no party p, integer $1 \leqslant \ell \leqslant k$, and voter group S jointly approving p such that $|S| \geqslant \ell\, n/k$ and $s(S) < \ell$. We show how to check in polynomial time for a fixed party p, whether there exists such a group of voters S. Then, this check can be repeated for every party contained in P in order to check if W satisfies PJR, which in total still requires polynomial time.

In the algorithm, submodular optimization will be utilized. Fix a party $p \in P$, and let N_p be the set of all voters who approve p. We define a function $f : 2^{N_p} \to \mathbb{R}$ such that

$$f(S) = s(S) - |S|\,\frac{k}{n}.$$

As the sum of a (submodular) coverage function s and a linear function, f is submodular. By multiplying f by n, we obtain an integer-valued function; thus, we can minimize f in polynomial time.

Intuitively, $s(S)$ is the number of seats the voters in S are represented by in W. To satisfy PJR, every coherent voter group of size $\ell\, n\, k$ has to be represented by at least ℓ seats. Thus, $\lfloor |S|\, k/n \rfloor$ is the number of seats S *should* be represented by according to PJR. Consequently, $f(S) = -1$ means that the voter group S is represented by one less seat than it should be in order to satisfy PJR. Formally, any $S \subseteq N_p$ is the witness of a PJR violation iff $f(S) \leqslant -1$, which we show in the following.

"$\Rightarrow$": Assume that $S \subseteq N_p$ shows a violation of PJR, that is $|S| \geqslant \ell\, n/k$ and $s(S) < \ell$ for some integer ℓ. Then, $s(S) \leqslant \ell - 1$ and $|S|\, k/n$ must be at least ℓ. This implies $f(S) \leqslant \ell - 1 - \ell = -1$.

"$\Leftarrow$": Fix some $S \subseteq N_p$ with $f(S) \leqslant -1$. It follows that $s(S) \leqslant |S|\, k/n - 1$. This directly implies that $s(S) < \lfloor |S|\, k/n \rfloor$. This shows a violation of PJR.

All in all, the procedure to check PJR can be summarized as follows: Iterate over every party $p \in P$. For each party, minimize the function f in polynomial time. If any function f has a minimal value $f(S) \leqslant -1$, then W violates PJR. If the minimal value of f is always greater than -1, then W satisfies PJR. ☐

[12]For a proof, see the paper by Aziz *et al.* (2018). Note that the paper shows that LS-PAV fulfills EJR when $\epsilon' = \frac{n}{k^2}$. Since $\epsilon \leqslant \epsilon'$ for all $k \geqslant 2$, their result carries over to LS-PAV with ϵ.