Computing Lindahl Equilibrium for Public Goods with and without Funding Caps

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Lindahl equilibrium is a solution concept for allocating a fixed budget across several divisible public goods. It always lies in the core, meaning that the equilibrium allocation satisfies desirable stability and proportional fairness properties. We consider a model where agents have separable linear utility functions over the public goods, and the output assigns to each good an amount of spending, summing to at most the available budget.

In the uncapped setting, each of the public goods can absorb any amount of funding. In this case, it is known that Lindahl equilibrium is equivalent to maximizing Nash social welfare, and this allocation can be computed by a public-goods variant of the proportional response dynamics. We introduce a new convex programming formulation for computing this solution and show that it is related to Nash welfare maximization through duality and reformulation. We then show that the proportional response dynamics is equivalent to running mirror descent on our new formulation, thereby providing a new and very immediate proof of the convergence guarantee for the dynamics. Our new formulation has similarities to Shmyrev's convex program for Fisher market equilibrium.

In the capped setting, each public good has an upper bound on the amount of funding it can receive, which is a type of constraint that appears in fractional committee selection and participatory budgeting. In this setting, existence of Lindahl equilibrium was only known via fixed-point arguments. The existence of an efficient algorithm computing one has been a long-standing open question. We prove that our new convex program continues to work when the cap constraints are added, and its optimal solutions are Lindahl equilibria. Thus, we establish that Lindahl equilibrium can be efficiently computed in the capped setting. Our result also implies that approximately core-stable allocations can be efficiently computed for the class of separable piecewise-linear concave (SPLC) utilities.

Contents

1 Introduction

We consider a setting where a fixed budget $B > 0$ needs to be spent on m divisible public goods. Thus, an outcome is a vector $x = (x_1, \dots, x_m) \in \mathbb{R}^m_{\geq 0}$ summing to at most B. Some of the public goods may additionally have *caps*, i.e., upper bounds on the amount of funding they can receive. How to distribute the spending across the goods is decided based on the preferences of n agents. We will consider agents with separable linear utility functions over the goods. Agents may have heterogeneous weights (which can be interpreted as endowments). We will study the solution concept of *Lindahl equilibrium*, which is based on a virtual market with personalized prices [Foley, 1970]. This equilibrium notion is known to lead to allocations that are fair to voters, formalized via the concept of the core from cooperative game theory [Fain et al., 2016].

The classic economics literature on public goods, starting with Samuelson [1954], focusses on how to arrive at the socially efficient amount of spending in the face of free-riding incentives. In contrast, we consider a fixed budget and are mostly concerned with how to divide it between different public goods. This approach, sometimes called *portioning* or *fair mixing*, has received increasing attention in computer science over recent years [see, e.g., Airiau et al., 2023, Aziz et al., 2020, Brandl et al., 2021, Fain et al., 2016], due to its many concrete applications. These include *participatory budgeting*, a method used by many cities to let residents vote over how the government will spend a fixed part of its budget [Aziz and Shah, 2021, Rey and Maly, 2023], and *donation platforms*, where donors can influence the distribution of a fixed matching fund [Brandl et al., 2022, Brandt et al., 2024]. The model also captures *committee elections* (i.e., multiwinner voting) in its fractional version [Aziz et al., 2023a, Suzuki and Vollen, 2024], as well as the *cake sharing* problem [Bei et al., 2024]. Voting methods for the public goods model can also be used to settle small-stakes issues such as a lecturer letting students vote over the distribution of class time across topics, or a team to vote over the frequencies with which they will go to different lunch venues. Companies and non-profit organizations can use the principles derived in this model to decide how to fairly and efficiently divide resources among units or grantees.

In many of these applications, it is desirable to select an outcome that is *representative* of the voters, in that every agent has an equal influence on the overall spending (or an influence that is proportional to the weight assigned to the agent). This can be formalized as a group fairness guarantee. In particular, we can require that the spending distribution lie in the *core*, which means that no subset of voters can construct an alternative way of spending their endowments in a way that they all prefer. We know that a core outcome always exists thanks to Foley [1970], who gave a definition of what he called *Lindahl equilibrium* (because he was inspired by ideas of Lindahl [1919]), proved its existence, and showed that it always lies in the core. This result was introduced to the computer science literature by Fain et al. [2016].

In the setting where each public good has no cap on funding (we call this the *uncapped setting*), Fain et al. [2016] showed that Lindahl equilibrium is equivalent to the rule that maximizes Nash social welfare (i.e., the product of agent utilities). The Nash rule has its root in the Nash [1950] bargaining solution, and its objective function has attractive mathematical properties such as scale-freeness [Moulin, 2004]. The Nash rule as applied to the public goods model had already been discussed earlier and independently from Lindahl equilibrium due to its attractive group fairness properties [Bogomolnaia et al., 2005, Guerdjikova and Nehring, 2014]. The connection between Lindahl equilibrium and the Nash rule is convenient since the latter can be efficiently computed via a convex program reminiscent of the classic Eisenberg–Gale program [Eisenberg, 1961, Eisenberg and Gale, 1959] for computing a market equilibrium for private goods. In addition, Brandl et al. [2022] showed that the Nash rule can be computed by running a simple proportional response dynamics which converges to the Nash outcome. They pointed out that the same convex program had been considered in several unrelated contexts such as in the portfolio selection literature, where this dynamics had also been discovered and shown to converge [Cover, 1984]. While the dynamics converges rapidly in practice, a formal bound on the speed of convergence had not been established by 2022, with Li et al. [2018, page 11] noting that the "algorithm [of Cover] possesses a guarantee of convergence but [no] convergence rate."

In the *capped setting*, the Nash rule loses its fairness properties and is not equivalent to Lindahl equilibrium. In contrast, Lindahl equilibrium retains its fairness properties, and its existence is known via fixed-point theorems [Foley, 1970]. However, this existence result only applies to strictly monotonic utility functions and thus does not allow agents to have valuations equal to 0 for some goods, and it does not allow for caps except through approximating them through appropriate 'saturating' utility functions [Fain et al., 2016, Munagala et al., 2022b]. Most importantly, the existence result is not algorithmic, and how to compute a Lindahl equilibrium was an open question. Fain et al. [2016] asked: "Is computing the Lindahl equilibrium for public goods computationally hard or is there a polynomial time algorithm even [when the public goods are capped]?"

Since then there has been no progress on this question. Indeed, Jiang et al. [2020] again noted that "we do not know how to compute the Lindahl equilibrium efficiently". It was even open how to compute any allocation that lies in the core, not necessarily a Lindahl equilibrium allocation. Cheng et al. [2020] noted that "it is not known how to compute such a core outcome efficiently even for [...] approval set utilities", and Suzuki and Vollen [2024] concluded that "there is no known polynomial time algorithm for computing fractional core". The need for a practical algorithm was particularly pressing in the work of Munagala et al. [2022b] who studied *indivisible* public goods and were aiming for allocations that are approximately in the core. Their best result is based on rounding a Lindahl equilibrium allocation and "yields a 9.27-core, though we do not know how to implement the resulting algorithm in polynomial time". To obtain a poly-time result, Munagala et al. [2022b] needed to avoid Lindahl equilibrium and in that case only achieved a 67.37-approximation.

1.1 Contributions

In the uncapped setting, we prove that the proportional response dynamics converges to a Lindahl equilibrium at a rate of $\log(nm)/t$. We show this by developing a new convex program, distinct from the standard Eisenberg–Gale-style program for Nash welfare maximization, and show that applying mirror descent to this program is equivalent to the proportional response dynamics, thereby allowing us to obtain the convergence rate from known results about mirror descent.[1] Our new convex program is related to the Eisenberg–Gale-style program through double duality: we show that it can be obtained by taking the dual, introducing new redundant variables, making a change of variable, and performing another dual derivation on this reformulated dual.

The duality and mirror descent relationship that we discover for public goods mirrors existing relationships known in the literature on *private goods* allocation using Fisher market equilibrium. For the private-good setting, equilibrium is also equivalent to maximizing Nash welfare. An alternative convex program for this equilibrium was developed by Shmyrev [1983, 2009]. A proportional response dynamics exists for the private goods case as well [Wu and Zhang, 2007, Zhang, 2011], and Birnbaum et al. [2011] showed that it is equivalent to mirror descent on the Shmyrev program. Our new program for the uncapped public goods setting is "Shmyrev-like" in its structure. A comparison of these convex programs is shown in Figure 1. We think it is surprising that it is

[1] While writing the paper, we became aware that Zhao [2023] has recently obtained the same convergence rate bound of $\log(nm)/t$. He obtained the convergence rate via a direct first-order analysis of the multiplicative gradient (MG) method. Zhao notes that "the extraordinary numerical performance of the MG method is rather surprising and somewhat mysterious [because it] is extremely simple". Our results demystify the performance of the dynamics, by showing that it is equivalent to mirror descent, but on a different convex program.

Private goods

$$\max_{x \geq 0} \sum_{i \in N} B_i \log \langle v_i, x_i \rangle \quad \text{s.t.} \sum_{i \in N} x_{ij} \leq 1 \quad \text{for all } j \in M$$

(a) The Eisenberg–Gale program.

$$\max_{b \geq 0, p \geq 0} \sum_{i \in N, j \in M} b_{ij} \log v_{ij} - \sum_{j \in M} p_j \log p_j$$
$$\text{s.t.} \sum_{i \in N} b_{ij} = p_j \quad \text{for all } j \in M$$
$$\sum_{j \in M} b_{ij} = B_i \quad \text{for all } i \in N$$

The final allocation is obtained as $x_{ij} = b_{ij}/p_j$.

(b) The Shmyrev program.

Public goods

$$\max_{x \geq 0} \sum_{i \in N} B_i \log \langle v_i, x \rangle \quad \text{s.t.} \sum_{j \in M} x_j \leq B$$

(c) The Nash welfare program.

$$\max_{b \geq 0, x \geq 0} \sum_{i \in N, j \in M_i} b_{ij} \log v_{ij} - \sum_{i \in N, j \in M_i} b_{ij} \log \frac{b_{ij}}{x_j}$$
$$\text{s.t.} \sum_{j \in M_i} b_{ij} = B_i \quad \text{for all } i \in N$$
$$\sum_{i \in N_j} b_{ij} = x_j \quad \text{for all } j \in M$$

(d) Our program.

Fig. 1. Comparison of convex programs for computing Fisher market equilibrium (private goods) and Lindahl equilibrium (public goods) for instances with linear utility functions.

possible to establish these analogous results, since there are important structural differences in the convex programs. For example, for private goods, the convex programs always have rational optimal solutions, while for public goods they may be irrational. In addition, for private goods, the Eisenberg–Gale program has the same number of variables (nm) as the Shmyrev program (after eliminating redundant price variables), while for public goods the Eisenberg–Gale-style program has m variables whereas our new convex program has nm variables. This dimensional difference means that new insights are required for transforming one program into the other through duality. Finally, the nonlinear term in the objective is a somewhat unusual normalized entropy function in our program, whereas it is the usual entropy on the prices in the private goods case.

In the capped setting, we answer the open problem raised by Fain et al. [2016] positively: Lindahl equilibrium can be computed efficiently in the capped setting. Indeed, the caps can be naturally added as constraints to our new convex program, and the resulting program correctly computes a Lindahl equilibrium respecting the caps. (In contrast, as is well-known, adding caps as constraints to the Nash welfare program does not lead to a Lindahl equilibrium.) This also establishes the existence of exact Lindahl equilibrium with caps, without approximations. Using our convex program, a Lindahl equilibrium can be computed in polynomial time using the ellipsoid method (to any desired accuracy), and it can be computed efficiently in practice by standard solvers with support for exponential cones, such as MOSEK. We present numerical experiments on real-life data from participatory budgeting, showing that solving our program is feasible even for large instances.

Finally, we show how to apply our result to compute approximately core-stable allocations for a broader class of utility functions, namely separable piecewise-linear concave utilities (SPLC).

1.2 Related Work

Lindahl equilibrium. Lindahl equilibrium was introduced by Foley [1970], who named this equilibrium concept after Lindahl [1919] who put forward related ideas of personalized taxation. However, note that there are other distinct ways of formalizing Lindahl's ideas [see van den Nouweland, 2015], including ratio and cost share equilibrium [Kaneko, 1977, Mas-Colell and Silvestre, 1989]. In this work, we use the Foley definition.

Uncapped setting. Our interest in Lindahl equilibrium is motivated mainly by their proportional fairness properties (notably the core). Such fairness properties have been studied in many related models, notably the "fair mixing" or "portioning" models [Airiau et al., 2023, Aziz et al., 2020, Bogomolnaia et al., 2005, Brandl et al., 2021, Fain et al., 2016, Gul and Pesendorfer, 2020] that correspond to what we call the uncapped setting. In this setting, Lindahl equilibrium coincides with the maximum Nash welfare solution which has been axiomatically characterized [Guerdjikova and Nehring, 2014] and noted for its strong participation incentives [Brandl et al., 2022] as well as its lowest-possible price of fairness [Michorzewski et al., 2020]. The Nash solution is also well-known to provide fair outcomes in other models, such as for private goods [Caragiannis et al., 2019].

Capped setting. What we call the capped setting has also been studied in various special cases under various names, such as cake sharing [Bei et al., 2024], fractional committee elections [Pierczyński and Skowron, 2022, Suzuki and Vollen, 2024], or divisible participatory budgeting [Aziz and Shah, 2021, Fain et al., 2016]. These works have mostly not considered Lindahl equilibrium, since there was no known way of computing one.

Discrete models. In discrete models, the public goods can either be fully funded or not at all. This model captures the way many cities run their participatory budgets, and has thus been well-studied including via core-like fairness notions such as EJR [Peters et al., 2021a, Rey and Maly, 2023], that were developed in the large literature on approval-based committee elections [Aziz et al., 2017, Lackner and Skowron, 2023, Peters, 2025]. There also exist proposals for definitions of Lindahl equilibrium for discrete models [Munagala et al., 2022a, Peters et al., 2021b].

Computation. In the uncapped setting, the maximum Nash welfare solution (and thus Lindahl equilibrium) can be efficiently computed via an Eisenberg–Gale-style convex program. This program has a simple structure (maximizing a natural objective function over the standard simplex), and Zhao [2023] has cataloged its appearance in many unrelated areas, including portfolio selection for maximizing log investment returns [Cover, 1984], information theory [Csiszár, 1974] and statistics [Vardi and Lee, 1993], and in medical imaging for positron emission tomography [Vardi et al., 1985]. Cover [1984] proposed a dynamics converging to the optimal solution of this program. Convergence proofs were also given by Csiszár [1984] and Brandl et al. [2022]. Later, Zhao [2023] obtained a convergence rate of $\log(nm)/t$ for this dynamics. This is the same rate that we establish, though his approach does not connect the dynamics to mirror descent. In the capped setting, very little was known about computation, except for a heuristic algorithm proposed by Fain et al. [2016] that worked well in their experiments.

Donor coordination. An important application of the public goods allocation problem we study is *donor coordination*, where a collection of donors wish to coordinate their charitable spending, for example by pooling their donations and voting over the division of the pool between different causes. Brandl et al. [2022] have argued (using an uncapped model with linear utilities) that the Nash rule and thus Lindahl equilibrium is a good solution concept for this use case [see also Greaves and Cotton-Barratt, 2023]. However, a key reason for coordinating donations is the potential presence of caps: some charities may have a limited "room for more funding". This issue is frequently discussed within Effective Altruism, citing cases similar to Example 5 [Peters, 2019]. As our work shows, the Nash solution does not extend well to settings with limited room for more funding, but Lindahl equilibrium as computed by our new convex program does. Brandt et al. [2024] also discuss Lindahl equilibrium applied to donor coordination, using Leontief utility functions.

Other applications of the core. The core has been employed as a fairness property in many other models. For example, Chaudhury et al. [2022] apply it to federated learning.[2] It has also been used for clustering [Caragiannis et al., 2024, Chen et al., 2019, Kellerhals and Peters, 2024], peer reviewer assignments [Aziz et al., 2023b], and sortition [Ebadian and Micha, 2025].

2 Setup

Let M be a set of m *public goods*, which we sometimes refer to as *projects*. We have an overall *budget* $B > 0$ that we can spend on the public goods. Let $N = \{1, \dots, n\}$ be a set of n *agents*. Each agent $i \in N$ has an individual budget $B_i > 0$ representing i's weight or endowment. These sum to the overall budget, $\sum_{i \in N} B_i = B$. In many applications, the entitlements are equal: $B_i = B/n$. Each agent i has a valuation $v_{ij} \geq 0$ for each public good $j \in M$. We write $v_i = (v_{ij})_{j \in M}$ for the vector of i's valuations. The *utility* of an agent $i \in N$ for an outcome $x \in \mathbb{R}^m_{\geq 0}$ is $u_i(x) = \langle v_i, x \rangle = \sum_{j \in M} v_{ij} x_j$. Thus, we use separable linear utilities. We write $M_i = \{j \in M : v_{ij} > 0\}$ for the projects that agent $i \in N$ likes, and we write $N_j = \{i \in N : v_{ij} > 0\}$ for the agents that support project $j \in M$.

In the *uncapped public goods* setting, an *allocation* is a vector $x = (x_j)_{j \in M}$ with $x_j \geq 0$ for all $j \in M$ and $\sum_{j \in M} x_j \leq B$. Here, x_j denotes the total spending on project b. In this definition, the public goods have no upper bound on how much of them we can spend on them, so in principle the entire budget B could be spent on a single good.

In the *capped public goods* setting, we add the additional constraint that each good $j \in M$ has a maximum amount $\text{cap}_j > 0$ that can be spent on it. Thus, in this setting, an *allocation* is a vector $x = (x_j)_{j \in M}$ with $0 \leq x_j \leq \text{cap}_j$ for all $j \in M$ and $\sum_{j \in M} x_j \leq B$. We assume that $\sum_{j \in M} \text{cap}_j \geq B$ (if not then we simply fully fund all the goods).

2.1 Lindahl Equilibrium

Our goal is to find a *Lindahl equilibrium* which is known to yield a fair and efficient allocation of public goods, in the sense that it yields an allocation that is Pareto efficient and lies in the (weak) core [Fain et al., 2016, Foley, 1970]. Let $p = (p_{ij})_{i \in N, j \in M}$ be a collection of non-negative *personalized prices*, with $p_{ij} \geq 0$ denoting the price that agent i needs to pay per unit of project j, and $p_i = (p_{ij})_{j \in M}$ denoting the vector of prices facing i.

Definition 1 (Lindahl Equilibrium). *Let x be an allocation and let p be a collection of non-negative personalized prices. Then (x, p) is a* Lindahl equilibrium *if*

- *x is* affordable*: we have $\langle p_i, x \rangle \leq B_i$ for every $i \in N$,*
- *x is* utility-maximizing*: for every $i \in N$ and every $y \in \mathbb{R}^m_{\geq 0}$ such that $0 \leq y_j \leq \text{cap}_j$ for all $j \in M$ and such that $\langle p_i, y \rangle \leq B_i$, we have $u_i(x) \geq u_i(y)$,*
- *x is* profit-maximizing*: for every $j \in M$, we have $\sum_{i \in N} p_{ij} \leq 1$, and whenever $x_j > 0$ then $\sum_{i \in N} p_{ij} = 1$.*

[2] Chaudhury et al. [2022, Theorem 2] show that the Nash rule satisfies core stability for arbitrary concave utilities. On first sight, this is in contradiction to our claim that Nash fails the core in the capped setting (Example 4). The difference is that Chaudhury et al. [2022] use a much weaker notion of the core which involves scaling utilities by the coalition size, rather than scaling the endowment. Fain et al. [2018, Section 1.6] discuss the distinction between these concepts.

We say that an allocation x is a *Lindahl equilibrium allocation* if there exist prices p such that (x, p) is a Lindahl equilibrium.

The distinctive property of a Lindahl equilibrium is that prices are personalized, but every agent demands the exact same bundle x of public goods. That is the content of the utility maximization condition: it says that every agent i can afford x given the prices $(p_{ij})_{j\in M}$ and i's budget B_i, and prefers x among all affordable allocations satisfying the cap constraint.

The interpretation of the profit maximization condition is less clear. Its most important effect is that it imposes some amount of efficiency: an equilibrium can only spend a positive amount of budget on projects that have the maximum total price (and generally prices are higher if agent valuations for the project are higher). The condition can be seen as "profit maximization" if we imagine that there is a central producer of the public goods who takes in money from the agents and produces the public goods (at a cost of 1 unit of money for 1 unit of public good). This interpretation is made formal in the following simple observation.

Lemma 1. *Let x be an allocation and let p be a collection of non-negative personalized prices. Then the following are equivalent:*

(a) for every $j \in M$, we have $\sum_{i\in N} p_{ij} \leq 1$, and whenever $x_j > 0$ then $\sum_{i\in N} p_{ij} = 1$,
(b) for every $y \in \mathbb{R}^m_{\geq 0}$, we have

$$\sum_{j\in M}\sum_{i\in N} p_{ij}x_j - \sum_{j\in M} x_j \geq \sum_{j\in M}\sum_{i\in N} p_{ij}y_j - \sum_{j\in M} y_j.$$

Proof. If the prices satisfy (a), then $\sum_{j\in M}\sum_{i\in N} p_{ij}x_j - \sum_{j\in M} x_j = \sum_{j\in M} x_j - \sum_{j\in M} x_j = 0$ and for every $y \in \mathbb{R}^m_{\geq 0}$, we have $\sum_{j\in M}\sum_{i\in N} p_{ij}y_j - \sum_{j\in M} y_j \leq \sum_{j\in M} y_j - \sum_{j\in M} y_j = 0$, establishing (b).

If the prices satisfy (b), but there is some $j \in M$ with $\sum_{i\in N} p_{ij} > 1$, then the profit attained by $y \in \mathbb{R}^m_{\geq 0}$ is unbounded as $y_j \to \infty$, a contradiction. If the prices satisfy (b), but there is some $j \in M$ with $x_j > 0$ but $\sum_{i\in N} p_{ij} < 1$, then taking y to be identical to x but with $y_j = 0$ gives larger profit, a contradiction. These two contradictions establish (a). □

Note that in condition (b), the producer compares x to every other possible vector $y \in \mathbb{R}^m_{\geq 0}$, even if y violates the cap-constraints or the overall budget constraint. Condition (b) is usually used as part of the definition of Lindahl equilibrium, but we have used condition (a) in our definition because it is simpler and more useful in proofs.

Example 1 (Personal projects). *Consider the uncapped setting, and suppose that each agent likes exactly one project that nobody else likes, so we have $N = M$, with $v_{ii} = 1$ for each $i \in N$ and $v_{ij} = 0$ for all $i \neq j$. In a Lindahl equilibrium, for each $i \in N$, utility maximization requires $x_i > 0$ and that the entire endowment B_i is spent on the personal project, so $p_{ii} = B_i/x_i$ and $p_{ij} = 0$ when $i \neq j$. By profit maximization, since $x_i > 0$, we get that $p_{ii} = 1$. Thus, $B_i/x_i = 1$ and so $x_i = B_i$. Therefore, there is a unique Lindahl equilibrium allocation x with $x_i = B_i$ for each $i \in N$.*

Every Lindahl equilibrium (x, p) can be *decomposed*: For each $i \in N$ and $j \in M$, write

$$b_{ij} = p_{ij}x_j$$

for the *contribution* of i towards j. This is a decomposition of x (similar to a notion considered by Brandl et al. [2022, Definition 2]) because the values $(b_{ij})_{ij}$ satisfy the following conditions:

- For each $j \in M$, we have $x_j = \sum_{i\in N} b_{ij}$. (This is trivial if $x_j = 0$ and if $x_j > 0$ it follows because $\sum_{i\in N} p_{ij} = 1$.)
- For each $i \in N$, we have $\sum_{j\in M} b_{ij} \leq B_i$. (This is simply a restatement of the affordability condition of the definition of Lindahl equilibrium.)

With this interpretation, we can see that p_{ij} equals the fraction of spending on project j that is contributed by agent i. This interpretation also appears in the definition of *ratio equilibrium* [Kaneko, 1977] which is equivalent to Foley's Lindahl equilibrium in the simple model we consider: We take the spending on a public good to be the same as the amount of the public good that is provided, which implies constant returns to scale, where several public goods equilibrium notions coincide [see also Mas-Colell and Silvestre, 1989, Moore, 2006, van den Nouweland, 2015].

Foley [1970] proved the existence of Lindahl equilibrium using a fixed-point theorem, in a model that is more general than ours. However, his result only applies to strictly monotonic preferences, and thus only establishes existence when $v_{ij} > 0$ for all $i \in N$ and $j \in M$. We will allow $v_{ij} = 0$. In the presence of zeros, it makes sense to consider Lindahl equilibria (x, p) that are what we call zero-respecting.

Definition 2 (Zero-respecting). *A Lindahl equilibrium (x, p) is* zero-respecting *if for all $i \in N$ and $j \in M$, whenever $v_{ij} = 0$ and $x_j > 0$ then $p_{ij} = 0$.*

This is a natural condition in view of the decomposition we considered above, because in a zero-respecting Lindahl equilibrium, an agent contributes only to projects with positive utility: if $v_{ij} = 0$ then $b_{ij} = 0$. This condition is also imposed in the decomposability condition of Brandl et al. [2022, Definition 2].

The following example shows that not every Lindahl equilibrium is zero-respecting, and that zero-respecting Lindahl equilibria may violate Pareto efficiency. This will motivate imposing a certain sufficient condition introduced below that will avoid this result.

Example 2 (Lindahl equilibrium may underspend). *Consider the following instance:*

	B_i	Project 1	Project 2
Agent 1	0.5	1	0
Agent 2	0.5	0	1
cap_j		0.25	∞

On this instance, the unique zero-respecting Lindahl equilibrium allocation is $x = (0.25, 0.5)$. To see this, note that each agent will demand the project that the agent likes, no matter the prices. Thus $x_1, x_2 > 0$. By profit maximization and the zero-respecting condition, we have $p_{11} = p_{22} = 1$ and $p_{12} = p_{21} = 0$. Then by the affordability and utility maximization conditions of Lindahl equilibrium, we get $x = (0.25, 0.5)$. Note that the total spending in this instance is 0.75, strictly less than the available budget of $B = 0.5 + 0.5$. In particular, x is Pareto-dominated by the allocation $y = (0.25, 0.75)$.

If we remove the zero-respecting condition, there exist other Lindahl equilibria. In particular, $x' = (0.25, 0.75)$ forms an equilibrium with the prices $p_1 = (1, \frac{1}{3})$ and $p_2 = (0, \frac{2}{3})$.[3]

Note that the formal model of Foley [1970] does not directly support caps, but these can be simulated ε-approximately through concave utility functions [Munagala et al., 2022b, Footnote 2]. We will be able to handle caps without any approximations.

2.2 Pareto-Optimality and the Core

Next we discuss how the Lindahl equilibrium relates to Pareto optimality and the set of allocations that are in the core. In the uncapped setting with strictly increasing valuations, the relationship between these concepts is straightforward, and was already studied by Foley [1970]. However, as

[3] Suppose we replace zero-valuations by $\varepsilon > 0$, i.e., we set $v_{12} = v_{21} = \varepsilon$. Then for all $\varepsilon > 0$, every Lindahl equilibrium allocation has $x_2 \geq 0.75$ by Corollary 1 (Pareto optimality). But for $\varepsilon = 0$, in a zero-respecting Lindahl equilibrium, $x_2 = 0.5$. Thus, Lindahl equilibrium does not necessarily converge to a zero-respecting Lindahl equilibrium as $\varepsilon \longrightarrow 0$.

we shall see, there is more nuance in the capped setting and in the presence of valuations equal to 0. We begin by introducing a sufficient condition that excludes examples like Example 2 where intuitively the caps of projects that receive non-zero valuations are too low. We will see that under this sufficient condition, every zero-respecting Lindahl equilibrium spends the entire budget, is Pareto efficient, and lies in the core.

For every $i \in N$, write $F_i = \{f \in N \mid M_i \cap M_f \neq \emptyset\}$ for the set of "friends" of i who agree that at least one common project has a positive valuation.

Definition 3. *An instance is* cap-sufficient *if we have* $\sum_{j \in M_i} \text{cap}_j \geq \sum_{f \in F_i} B_f$ *for all* $i \in N$.

There are many interesting settings in which instances are always cap-sufficient, including:

- The uncapped setting where $\text{cap}_j = +\infty$ for all $j \in M$.
- All valuations are positive: $v_{ij} > 0$ for all $i \in N$ and $j \in M$. (Proof: In this case, $M_j = M$ and $F_i = N$, so the cap-sufficiency condition is implied by our general assumption that $\sum_{j \in M} \text{cap}_j \geq B$.)
- Each agent has positive utility for goods whose total cap reaches the budget: $\sum_{j \in M_i} \text{cap}_j \geq B$.

We will show that Lindahl equilibrium has particularly desirable properties on cap-sufficient instances. A key consequence of cap-sufficiency is that every voter spends their entire budget.

Proposition 1. *On a cap-sufficient instance, if* (x, p) *is a zero-respecting Lindahl equilibrium, then*

(i) for every $i \in N$*, we have* $\langle p_i, x \rangle = B_i$*,*
(ii) we have $\sum_{j \in M} x_j = B$*,*
(iii) for every $i \in N$ *and every* $y \in \mathbb{R}^m_{\geq 0}$ *such that* $0 \leq y_j \leq \text{cap}_j$ *for all* $j \in M$ *and such that* $\langle v_i, y \rangle \geq \langle v_i, x \rangle$*, we have* $\langle p_i, y \rangle \geq B_i$*.*

Proof. (i) Suppose for a contradiction that $\langle p_i, x \rangle < B_i$. We claim that then for all $j \in M_i$, we have $x_j = \text{cap}_j$: Otherwise, if $x_j < \text{cap}_j$, we can increase x_j, thereby increasing the utility of i, and a sufficiently small increase is affordable since i does not spend all of B_i, contradicting utility maximization.

Now, because (x, p) is zero-respecting, for each $j \in M_i$, only friends of i will contribute to j because $N_j \subseteq F_i$. Thus, $b_{i'j} = 0$ if $i' \in N \setminus F_i$. But then

$$\sum_{j \in M_i} \text{cap}_j = \sum_{j \in M_i} x_j = \sum_{j \in M_i} \sum_{f \in F_i} b_{fj} < \sum_{f \in F_i} B_f,$$

contradicting that the instance is cap-sufficient.

(ii) Using (i), we deduce that

$$\sum_{j \in M} x_j = \langle 1, x \rangle = \sum_{j \in M} \sum_{i \in N} p_{ij} x_j = \sum_{i \in N} \sum_{j \in M} p_{ij} x_j = \sum_{i \in N} B_i = B.$$

(iii) For a contradiction, suppose there is $i \in N$ and a cap-respecting allocation y such that $\langle v_i, y \rangle \geq \langle v_i, x \rangle$ but with $\langle p_i, y \rangle < B_i$. Due to (i), we have $\langle p_i, y \rangle < \langle p_i, x \rangle$. Thus, there exists some $j \in M$ such that $p_{ij} > 0$ and $y_j < x_j$. Since (x, p) is zero-respecting, we have $v_{ij} > 0$. Now consider an allocation obtained from y but with the j-coordinate increased by a small amount. For a small enough increase, the resulting allocation respects the cap-constraints (because $y_j < x_j \leq \text{cap}_j$) and is affordable for i (because $\langle p_i, y \rangle < B_i$), but gives i strictly higher utility, contradicting the utility-maximization condition of Lindahl equilibrium. (Note that the constructed allocation may not respect the overall budget constraint, but that is not required by the definition of Lindahl equilibrium.) □

A major reason to be interested in Lindahl equilibrium is that it always lies in the weak core, which is a fairness or stability property formalizing proportional representation.

Definition 4 (Core). *An allocation x is in the* core *if there is no "blocking coalition" $S \subseteq N$ and no objection $z = (z_j)_{j \in M} \in \mathbb{R}^m_{\geq 0}$ with $0 \leq z_j \leq \mathrm{cap}_j$ for all $j \in M$, such that $\sum_{j \in M} z_j \leq \sum_{i \in S} B_i$ (it can be afforded by the blocking coalition) and for all $i \in S$, we have $\langle v_i, z \rangle \geq \langle v_i, x \rangle$ (every coalition member weakly prefers the objection) and the inequality is strict for at least one $i \in S$. It is in the* weak core *if there are no such S and z such that $\langle v_i, z \rangle > \langle v_i, x \rangle$ for all $i \in S$.*

Foley [1970, Section 6] proved that Lindahl equilibrium allocations are in the weak core, though his model implicitly assumed cap-sufficiency. We can more generally show the following.

Proposition 2. *Let (x, p) be a Lindahl equilibrium. Then x lies in the weak core. If the instance is cap-sufficient and (x, p) is zero-respecting, then x lies in the core.*

Proof. Suppose not, and suppose $S \subseteq N$ is a blocking coalition with objection z satisfying $\sum_{j \in M} z_j \leq \sum_{i \in S} B_i$. We now claim that $\sum_{i \in S} \langle p_i, z \rangle > \sum_{i \in S} B_i$.

- Under the assumption that x fails the weak core, note that since for every $i \in S$ we have $\langle v_i, z \rangle > \langle v_i, x \rangle$, the utility maximization condition of Lindahl equilibrium implies that $\langle p_i, z \rangle > B_i$. Summing over $i \in S$ establishes the claim.
- Under the assumption that x fails the core, that (x, p) is zero-respecting, and that the instance is cap-sufficient, Proposition 1(iii) implies that since $\langle v_i, z \rangle \geq \langle v_i, x \rangle$ for all $i \in S$, we have $\langle p_i, z \rangle \geq B_i$. Since we have $\langle v_i, z \rangle > \langle v_i, x \rangle$ for at least one $i \in S$, we have $\langle p_i, z \rangle > B_i$ for that i due to utility maximization. Again, summing over $i \in S$ establishes the claim.

Combining the claim with the non-negativity of prices and profit maximization, we have

$$\sum_{i \in S} B_i < \sum_{i \in S} \langle p_i, z \rangle \leq \sum_{i \in N} \langle p_i, z \rangle \leq \langle 1, z \rangle \leq \sum_{i \in S} B_i,$$

a contradiction. □

As a special case, taking $S = N$ in Proposition 2, we see that Lindahl equilibrium allocations are (weakly) Pareto efficient, establishing a version of the First Welfare Theorem.

Definition 5 (Pareto-optimality). *An allocation x is* Pareto-optimal *if there is no allocation x' such that $u_i(x') \geq u_i(x)$ for all $i \in N$ and $u_i(x') > u_i(x)$ for some $i \in N$. It is* weakly *Pareto-optimal if there is no x' with $u_i(x') > u_i(x)$ for all $i \in N$.*

Corollary 1. *Let (x, p) be a Lindahl equilibrium. Then x is weakly Pareto optimal. If the instance is cap-sufficient and (x, p) is zero-respecting, then x is Pareto optimal.*

As another special case of the core result, it is worth noting that Lindahl equilibria also gives guarantees for individual agents (by considering $S = \{i\}$), leading to an axiom generalizing the *individual fair share* property of Aziz et al. [2020].

Corollary 2. *Let x be a Lindahl equilibrium allocation, and let $i \in N$. Then for every allocation z with $\sum_{j \in M} z_j \leq B_i$ and $z_j \leq \mathrm{cap}_j$ for all $j \in M$, we have $u_i(x) \geq u_i(z)$.*

3 Convex Optimization Background

This section gives background on convex optimization and the mirror descent algorithm.

Basic definitions. Let $f : \mathbb{R}^n \to (-\infty, \infty]$ be a function. It is called *proper* if there exists $x \in \mathbb{R}^n$ with $f(x) < \infty$. Its *subdifferential* at $x \in \mathbb{R}^n$ is $\partial f(x) = \{g \in \mathbb{R}^n : f(y) \geq f(x) + \langle g, y - x \rangle \text{ for all } y \in \mathbb{R}^n\}$. Elements of $\partial f(x)$ are called *subgradients*. The *convex conjugate* of f is the function $f^* : \mathbb{R}^n \to [-\infty, \infty]$ defined by $f^*(y) = \sup_{x \in \mathbb{R}^n}(\langle y, x \rangle - f(x))$. We use the convention $0 \log 0 = 0$. We write $B \cdot \Delta^m = \{x \in \mathbb{R}^m_{\geq 0} : \sum_j x_j = B\}$ for the scaled simplex.

KKT optimality conditions. We will use the following version of the Karush–Kuhn–Tucker theorem, which also works for non-differentiable objective functions.

THEOREM 1 (RUSZCZYNSKI, 2011, THM 3.34). *Let x^* be an optimal solution to the program*

$$\text{minimize } f(x) \text{ subject to } h_i(x) \le 0 \text{ for } i = 1, \ldots, m$$

where $f : \mathbb{R}^n \to (-\infty, \infty]$ and $h_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$, are proper convex functions. Assume that f is continuous at some feasible point, and that Slater's constraint qualification is satisfied, so that there is a feasible point x with $h_i(x) < 0$ for $i = 1, \ldots, m$. Then there exist $\lambda_1, \ldots, \lambda_m \ge 0$ such that

$$0 \in \partial f(x^*) + \textstyle\sum_{i=1}^m \lambda_i \partial h_i(x^*) \quad \text{and} \quad \lambda_i h_i(x^*) = 0 \text{ for } i = 1, \ldots, m.$$

Conversely, if x^* satisfies the constraints $h_i(x^*) \le 0$ for $i = 1, \ldots, m$ and there exist $\lambda_1, \ldots, \lambda_m \ge 0$ satisfying the above conditions, then x^* is a global minimum.

The KKT theorem can be used to characterize the optimal solutions of convex programs. Two such optimization problems will appear repeatedly in our derivations, and so we state them here.

LEMMA 2. *(a) Let $y \in \mathbb{R}^n$. Suppose x^* minimizes $\sum_{i=1}^n x_i \log x_i - \langle y, x \rangle$ subject to $x \in \Delta^n$. Then $x_i^* = e^{y_i}/(\sum_{j=1}^n e^{y_j})$ for $i = 1, \ldots, n$.*
(b) Let $y \in \mathbb{R}^n_{\ge 0} \setminus \{0\}$. Suppose x^* maximizes $\sum_{i=1}^n y_i \log x_i$ subject to $x \in \Delta^n$. Then $x_i^* = y_i/(\sum_{j=1}^n y_j)$ for $i = 1, \ldots, n$.

PROOF. For (a), see the book by Beck [2017, Example 3.71]. For (b), let us assume that $y_i > 0$ for all i, since for i with $y_i = 0$, it is optimal to set $x_i = 0$ and we can ignore these indexes when optimizing the others. Under this assumption, note that any optimal solution must have $x_i > 0$ for all i. Applying the KKT theorem, this means that the multipliers of the non-negativity constraints are 0. Thus, the stationarity condition implies that $0 = -y_i/x_i + \lambda$, where $\lambda \ge 0$ is the multiplier for the constraint $\sum_i x_i = 1$. Thus, $x_i = \lambda y_i$. Summing over all i, we see that $\lambda = 1/(\sum_i y_i)$, which gives the result. □

Convex programming duality. We will use the following recipe for deriving the dual of convex programs with linear constraints. The recipe was explicitly given in Cole et al. [2017], though it is also a direct consequence of Fenchel duality [Rockafellar, 1970, Theorem 31.1], as we show for completeness .

THEOREM 2. *Let f be a proper convex function. The following programs are dual:*

- $\min_x f(x) - \langle c, x \rangle$ *subject to $Ax \ge b$,*
- $\max_{y,z} \langle b, z \rangle - f^*(y)$ *subject to $zA = y - c$ and $z \ge 0$.*

In particular, the two programs have the same objective value, provided that there is a point x with $f(x) < \infty$ and $Ax > b$.

PROOF. Consider the concave function

$$h(x) = \begin{cases} \langle c, x \rangle & \text{if } Ax \ge b, \\ -\infty & \text{otherwise.} \end{cases}$$

Its concave conjugate is $h_*(y) =_{\text{def}} \inf_x \langle y, x \rangle - h(x) = \inf_{x:Ax \ge b} \langle y, x \rangle - \langle c, x \rangle = \inf_{x:Ax \ge b} \langle y - c, x \rangle$. Now, applying Fenchel duality and LP duality, we have

$$\begin{aligned} \min_{x:Ax \le b} f(x) - \langle c, x \rangle &= \inf_x \; f(x) - h(x) \\ &= \sup_y \; h_*(y) - f^*(y) && \text{(Fenchel's duality theorem)} \end{aligned}$$

$$
\begin{aligned}
&= \sup_y \Big(\inf_{x\,:\,Ax \geq b} \langle y - c, x\rangle - f^*(y) \Big) \\
&= \sup_y \Big(\sup_{z \geq 0 : zA = y - c} \langle b, z\rangle - f^*(y) \Big) && \text{(LP duality)} \\
&= \sup_{y,z\,:\,z \geq 0,\, zA = y - c} \langle b, z\rangle - f^*(y),
\end{aligned}
$$

showing that the two programs are dual. Fenchel's duality theorem applies provided that f and h have domains whose relative interior intersects [Rockafellar, 1970, Theorem 31.1], which follows from the existence of some x with $f(x) < \infty$ and $Ax > b$. □

Mirror descent. The mirror descent (MD) algorithm is a first-order method for convex minimization which generalizes projected gradient descent to allow for more general notions of distance. Given a convex set X and a convex function f, the goal is to minimize f over X via first-order updates. MD relies on a Bregman divergence $D_h(x\|y)$, which is a convex function that measures the difference between x and y. The function D_h is constructed from some 1-strongly convex *reference function* h as $D_h(x\|y) = h(x) - h(y) - \langle \nabla h(y), x - y\rangle$. For example, taking the negative entropy reference function $h(x) = \sum_i x_i \log x_i$, the Bregman divergence becomes the KL divergence, $D_h(x\|y) = \sum_i x_i \log(x_i/y_i)$. The update rule for MD is

$$
x^{t+1} = \arg\min_{x \in X} \langle \nabla f(x^t), x\rangle + \frac{1}{\eta} D_h(x\|x^t), \tag{1}
$$

where $\eta > 0$ is a stepsize parameter. There are a variety of convergence results for MD. We will specifically be interested in the case where a special relationship holds between the objective f and the reference function h, knowns as *relative smoothness*. The function f is said to be 1-smooth relative to the reference function h when it holds for all $x, y \in \operatorname{relint} X$ that

$$
f(a) \leq f(b) + \langle \nabla f(b), a - b\rangle + D_h(a\|b).
$$

The following theorem from Birnbaum et al. [2011] shows that when the reference function h is chosen such that relative smoothness holds, the sequence of iterates generated by mirror descent converges at a rate of $O(1/t)$:

Theorem 3 (Birnbaum et al., 2011, Theorem 3). *Suppose that f is 1-smooth relative to the reference function h, and we run mirror descent using h as the distance-generating function. Let x^* be an optimal solution. Then the sequence of iterates generated by mirror descent satisfies:*

$$
f(x^t) - f(x^*) \leq \frac{D_h(x^*\|x^0)}{t}
$$

4 Uncapped Public Goods

We begin by analyzing the uncapped setting, and begin by characterizing the Lindahl equilibrium prices, which will be helpful for understanding the convex programs we discuss. Note that if (x, p) is a Lindahl equilibrium, then each agent will only demand public goods that maximize the "bang-per-buck" ratio v_{ij}/p_{ij}. (When $v_{ij} = 0$ and $x_j > 0$, the agent also demands project j but only if $p_{ij} = 0$. Thus, in the uncapped setting, every Lindahl equilibrium is zero-respecting.) Thus, the quantity v_{ij}/p_{ij} must be equal for all projects j with $x_j > 0$ and $v_{ij} > 0$. Therefore, $p_{ij} \propto v_{ij}$, say with factor of proportionality α. Because each agent spends their entire budget (Proposition 1 (i), which applies since in the uncapped setting Lindahl equilibrium is always zero-respecting), we have $B_i = \sum_{j \in M_i} p_{ij} x_j = \sum_{j \in M_i} \alpha v_{ij} x_j = \alpha \langle v_i, x\rangle$. Thus we deduce that in the uncapped setting,

$$
p_{ij} = B_i \cdot \frac{v_{ij}}{\langle v_i, x\rangle} \quad \text{for all } j \in M \text{ with } x_j > 0. \tag{2}
$$

(Note that when $v_{ij} = 0$ and $x_j > 0$, (2) just says $p_{ij} = 0$ which follows because (x, p) is zero-respecting, so (2) also holds for $v_{ij} = 0$.)

Now consider a project $j \in M$ with $x_j = 0$. Because i does not demand it, its bang-per-buck must be weakly below the bang-per-buck of funded projects. From (2), it then follows that

$$p_{ij} \geq B_i \cdot \frac{v_{ij}}{\langle v_i, x\rangle} \quad \text{for all } j \in M \text{ with } x_j = 0. \tag{3}$$

As we explained in Section 2.1, any Lindahl equilibrium can be decomposed into individual contributions $b_{ij} = p_{ij}x_j$. From (2), it follows that in Lindahl equilibrium,

$$b_{ij} = B_i \cdot \frac{v_{ij}x_j}{\langle v_i, x\rangle} \quad \text{for all } j \in M, \tag{4}$$

or more simply that $b_{ij} \propto v_{ij}x_j$, so contributions are proportional to the utility i obtains in x from j. One can view (4) as a kind of fixed-point property implied by Lindahl equilibrium [Guerdjikova and Nehring, 2014], and it suggests the proportional response dynamics that we will study later.

4.1 Nash Welfare and the Eisenberg–Gale Program

In the uncapped setting (i.e. $\text{cap}_j = +\infty$ for all $j \in M$), Lindahl equilibrium allocations can be nicely characterized as those maximizing the Nash social welfare $\prod_i u_i(x)$ [Fain et al., 2016]. Such an allocation can be computed by solving the following convex program:

$$\begin{aligned} \max_{x \geq 0} \ & \sum_{i \in N} B_i \log \langle v_i, x\rangle \\ \text{s.t.} \ & \sum_{j \in M} x_j \leq B \end{aligned} \tag{5}$$

This program is the public-goods analogue of the Eisenberg–Gale convex program for computing a Fisher market equilibrium with private goods [Eisenberg, 1961, Eisenberg and Gale, 1959]. Based on this description of the prices, we can now one can analyze the KKT conditions of Program 5 to show that it exactly computes Lindahl equilibrium.

Theorem 4. [Fain et al., 2016, Corollary 2.3] *In the uncapped setting, an allocation x is a Lindahl equilibrium allocation if and only if it is an optimal solution to Program 5.*

Proof. Let x be an optimal solution to Program 5. Note first that every agent has strictly positive utility at x since otherwise the objective value would be $-\infty$. Thus, the objective function is differentiable at x. By Theorem 1, there exists $\lambda \geq 0$ (corresponding to the budget constraint) and $(\mu_j)_{j\in M} \geq 0$ (corresponding to the non-negativity constraints) such that for every $j \in M$, we have $0 = -\sum_{i\in N} B_i \frac{v_{ij}}{\langle v_i,x\rangle} + \lambda - \mu_j$. Thus $\lambda \geq \sum_{i\in N} B_i \frac{v_{ij}}{\langle v_i,x\rangle}$, with equality if $x_j > 0$. Multiplying by x_j, summing over j, and rearranging, we get $\sum_{j\in M} \lambda x_j = \sum_{j\in M}\sum_{i\in N} B_i \frac{v_{ij}x_j}{\langle v_i,x\rangle}$. This simplifies to $\lambda B = \sum_{i\in N} B_i = B$, so $\lambda = 1$. Hence for each $j \in M$, we have

$$\sum_{i\in N} B_i \frac{v_{ij}}{\langle v_i, x\rangle} \leq 1, \text{ with equality if } x_j > 0.$$

Set $p_{ij} = B_i \frac{v_{ij}}{\langle v_i,x\rangle}$. Then (x, p) is a Lindahl equilibrium: The above inequality immediately establishes the profit maximization condition. For utility maximization, note that the "bang-per-buck" of project j to agent i is $v_{ij}/p_{ij} = \langle v_i, x\rangle/B_i$, which is constant, so that all allocations that use up all of the agent's budget are utility maximizing. Since $\langle p_i, x\rangle = B_i$, it follows that x is utility maxmizing for i.

Conversely, suppose (x, p) is a Lindahl equilibrium. Set $\lambda = 1$ and $\mu_j = 1 - \sum_{i\in N} B_i \frac{v_{ij}}{\langle v_i,x\rangle}$ for each $j \in M$. For complementary slackness, note that if $x_j > 0$, then from (2) we have $\mu_j = 1 - \sum_{i\in N_j} p_{ij}$, and thus by the profit maximization condition of Lindahl equilibrium, we get $\mu_j = 0$. Complementary slackness also holds for λ since $\sum_{j\in M} x_j = B$ by Proposition 1(ii). We also have $\mu_j \geq 0$, combining (2), (3) and the profit maximization condition of Lindahl equilibrium. Finally, it is easy to check stationarity; for every $j \in M$ we have

$$-\sum_{i\in N} B_i \frac{v_{ij}}{\langle v_i, x\rangle} + \lambda - \mu_j = -\sum_{i\in N} B_i \frac{v_{ij}}{\langle v_i, x\rangle} + \sum_{i\in N} B_i \frac{v_{ij}}{\langle v_i, x\rangle} = 0.$$

Thus, by Theorem 1, x is an optimal solution to Program 5. □

Fain et al. [2016, Theorem 2.2] also present Eisenberg–Gale-style programs for computing Lindahl equilibria for certain non-linear utility functions called "scalar separable non-satiating" including CES and Cobb-Douglas utilities.

Interestingly, for Fisher market equilibrium, the Eisenberg–Gale program always admits a rational solution [Devanur et al., 2008, Vazirani, 2012]. However, this is not the case in our public goods setting,[4] as the following example shows [see also Airiau et al., 2023, Theorem 5].

Example 3 (Irrational Lindahl equilibrium allocation). *Consider the uncapped setting with 4 agents with equal budgets $B_i = \frac{1}{4}$ and with three projects. The agents have the following valuations:*

	B_i	Project 1	Project 2	Project 3
Agent 1	0.25	1	0	0
Agent 2	0.25	1	0	1
Agent 3	0.25	1	1	0
Agent 4	0.25	0	1	1
cap_j		∞	∞	∞

By Theorem 4, a Lindahl equilibrium allocation x forms an optimal solution to Program 5. Since projects 2 and 3 are symmetric and the objective function of Program 5 is strictly convex, we have $x_2 = x_3$. Since $B = 1$, we deduce that $x_1 = 1 - 2x_2$. Thus, the objective function of Program 5 can be written as $\frac{1}{4}(\log(1-2x_2) + 2\log(1-x_2) + \log(2x_2))$. Exponentiating, this is equivalent to maximizing $(1-2x_2)(1-x_2)^2(2x_2)$. Setting its derivative to 0, we find that it has its unique maximum at $x_2 = \frac{1}{16}(7-\sqrt{17}) \approx 0.1798$. Thus, x is irrational and the unique Lindahl equilibrium allocation.

4.2 A New Convex Program

We will present a new convex program which also captures the Lindahl equilibrium concept in the uncapped setting. As we will see, this convex program will yield several useful results. First, we will use it to show that the proportional response dynamics for uncapped public goods can indeed be interpreted as mirror descent with the entropy distance, just as in the Fisher market setting. Secondly, extending this convex program will allow us to give the first computational results for the capped public goods setting. Our new convex program is in the spirit of the Shmyrev convex program for Fisher markets for private goods [Shmyrev, 1983, 2009], though there are important

[4]For Fisher markets, the proof sets up a system of linear inequalities whose variables correspond to the reciprocals of equilibrium prices, $1/p_j$. However, profit maximization in Lindahl equilibrium (which has no analogue in Fisher markets) involves the sum $\sum_i p_{ij}$ which is not linear in the reciprocals of prices. Thus, the Fisher market argument does not generalize.

differences. The convex program is as follows:

$$\begin{aligned} \max_{b\geq 0, x\geq 0} \quad & \sum_{i\in N, j\in M_i} b_{ij}\log v_{ij} - \sum_{i\in N, j\in M_i} b_{ij}\log\left(b_{ij}/x_j\right) \\ \text{s.t.} \quad & \sum_{j\in M_i} b_{ij} = B_i, \qquad \forall i\in N \\ & \sum_{i\in N_j} b_{ij} = x_j, \qquad \forall j\in M \end{aligned} \tag{6}$$

The program has two sets of variables, though one is implied by the other. The x_j variable has the same interpretation as in Program 5: it is the amount of budget allocated to project j. The b_{ij} variables can be interpreted as the share of agent i's budget B_i that they allocate towards project j. Note that each x_j variable is directly implied by the choice of the b_{ij} variables across agents i. It is only there as a convenience variable, and we could replace each occurrence of it in Program 6 by $\sum_{i\in N_j} b_{ij}$. Indeed, in our proofs, we will mostly work directly with this formulation that optimizes only over the b_{ij} variables.

To gain some intuition for Program 6, suppose that we already knew the optimum value of the x_j variables, and thus can treat them as constants and use Program 6 to merely compute the values of the b_{ij} variables. From Lemma 2 (a), these optimum values satisfy $b^*_{ij} \propto v_{ij}x_j$, which exactly matches the condition (4) that we derived earlier from the definition of Lindahl equilibrium.

While our program has some similarity to the Shmyrev program for private goods [Birnbaum et al., 2011, Shmyrev, 2009], it has the following important differences. First, the Shmyrev program contains variables corresponding to prices, which do not appear in our program. Second, the original primal variables x_j appear directly in our program, whereas in Shmyrev's program these are a non-linear function of the corresponding b variables. Third, we have a somewhat unusual term that looks like a partially-normalized entropy in our objective, whereas Shmyrev's program only requires using a typical negative entropy term over prices.

4.3 Connecting the Eisenberg–Gale and Shmyrev Programs via Duality

Program 5 and Program 6 can be related to each other through "double duality". We need the following lemma , which derives the convex conjugate of a convex function that will appear in the dual of Program 5.

Lemma 3. *Consider some $q \in \mathbb{R}^{n\times m}$ and let q_j be the j-th column of q. For $j \in M$, let $g_j(q_j) = \sum_{i\in N} e^{q_{ij}}$ and let $g(q) = B\cdot \max_{j\in M} g_j(q_j)$. Then the convex conjugate of g is*

$$g^*(\beta) = \sum_{ij} \beta_{ij}\log\frac{\beta_{ij}}{x_j(\beta)} - \beta_{ij},$$

where $x_j(\beta) = B\cdot(\sum_i \beta_{ij})/(\sum_{ij'} \beta_{ij'})$.

Proof. We compute the convex conjugate of g using standard formulas for the conjugate of a separable function, rescalings of a function, and the exponential function [see, e.g., Beck, 2017, Sections 4.3 and 4.4]. We also use Sion's minimax theorem [Komiya, 1988] which states that if X is convex, Y is convex and compact, and f is a real-valued function on $X\times Y$ that is convex in its first argument and concave in its second argument, then $\sup_{x\in X}\min_{y\in Y} f(x,y) = \min_{y\in Y}\sup_{x\in X} f(x,y)$. Finally, given $\beta = (\beta_{ij})_{i\in N, j\in M}$, we write $\beta_j = (\beta_{i,j})_{i\in N}$.

Putting all of this together, we derive that

$$g^*(\beta) = \sup_q \langle \beta, q\rangle - g(q)$$

$$= \sup_q \min_{j \in M} \langle \beta, q \rangle - B \cdot g_j(q_j) \qquad \text{(definition of } g\text{)}$$

$$= \sup_q \min_{x \in \Delta(B)} \langle \beta, q \rangle - \sum_{j \in M} x_j \cdot g_j(q_j) \qquad \text{(minimum attained at a vertex)}$$

$$= \min_{x \in \Delta(B)} \sup_q \langle \beta, q \rangle - \sum_{j \in M} x_j \cdot g_j(q_j), \qquad \text{(Sion's minimax theorem)}$$

$$= \min_{x \in \Delta(B)} \sum_{j \in M} \sup_{q_j} \langle \beta_j, q_j \rangle - x_j \cdot g_j(q_j), \qquad \text{(conjugate of a separable function)}$$

$$= \min_{x \in \Delta(B)} \sum_{j \in M} x_j \cdot g_j^*(\beta_j / x_j) \qquad ((\alpha f)^*(y) = \alpha f^*(y/\alpha))$$

$$= \min_{x \in \Delta(B)} \sum_{j \in M} x_j \cdot \left(\sum_{i \in N} \frac{\beta_{ij}}{x_j} \log \frac{\beta_{ij}}{x_j} - \frac{\beta_{ij}}{x_j} \right) \qquad (\exp^*(y) = y \log y - y)$$

$$= \min_{x \in \Delta(B)} \sum_{ij} \beta_{ij} \log \frac{\beta_{ij}}{x_j} - \beta_{ij}$$

$$= \sum_{ij} \beta_{ij} \log \frac{\beta_{ij}}{x_j(\beta)} - \beta_{ij}, \qquad \text{(Lemma 2(b))}$$

as required. □

Now we can state the result formalizing the relationship between the two programs: their dual programs are equivalent. This result is an analogue of a result for private goods, where the Shmyrev and the Eisenberg–Gale program also share a dual after reformulation [Cole et al., 2017].

Theorem 5. *Program 6 is the dual of the dual of the Eisenberg–Gale convex program for public goods, after reformulation.*

Proof. Using the Fenchel duality in Theorem 2, the dual of Program 5 is

$$\begin{aligned} \min_{\lambda \geq 0, \beta \geq 0} \; & B\lambda - \sum_{i \in N} B_i \log \beta_i \\ \text{s.t.} \; & \sum_{i \in N} \beta_i v_{ij} \leq \lambda, \forall j \in M \end{aligned} \tag{7}$$

Now we rewrite Program 7 by introducing a redundant set of variables λ_{ij}, which will represent the "bid" $\beta_i v_{ij}$ that agent i makes on project j. This gives the following program:

$$\begin{aligned} \min_{\lambda \geq 0, \beta \geq 0} \; & B\lambda - \sum_{i \in N} B_i \log \beta_i \\ \text{s.t.} \; & \sum_{i \in N} \lambda_{ij} \leq \lambda, \forall j \in M \\ & \beta_i v_{ij} \leq \lambda_{ij}, \forall i \in N, j \in M \end{aligned} \tag{8}$$

Having removing the summation from the bottom constraint in Program 7, we can now apply the logarithm to the bottom constraint of program (8) and get a separation into individual terms. We note that in optimum, the value of λ is the maximum over $j \in M$ of the value of $\sum_{i \in N} \lambda_{ij}$, so we can replace λ in the objective function by a maximum. In addition, we perform two changes of variable: we write $\gamma_i = -\log \beta_i$ and $q_{ij} = \log \lambda_{ij}$. Thereby we arrive at the following program:

$$\min_{q,\gamma} B \cdot \left(\max_{j\in M} \sum_{i\in N} e^{q_{ij}}\right) + \sum_{i\in N} B_i\gamma_i$$
$$\text{s.t. } \log v_{ij} \leq q_{ij} + \gamma_i, \forall i \in N, j \in M \tag{9}$$

Next we derive the dual of (9), again using the Fenchel dual from Theorem 2. For $j \in M$, let $g_j(q) = \sum_{i\in N} e^{q_{ij}}$ and let $g(q) = B\cdot\max_{j\in M} g_j(q)$. Write b_{ij} for the dual variables for the constraints in Program 9. Then the dual of (9) is

$$\max_{b\geq 0} \sum_{ij} b_{ij} \log v_{ij} - g^*(b)$$
$$\text{s.t. } \sum_{j\in M} b_{ij} = B_i,\ \forall i \in N \tag{10}$$

By Lemma 3, the term $g^*(b)$ equals $\sum_{ij} b_{ij} \log \frac{b_{ij}}{x_j(\beta)} - b_{ij}$, where $x_j(b) = B \cdot \sum_i b_{ij}/\sum_{ij'} b_{ij'}$. In Program 10, the denominator $\sum_{ij'} b_{ij'}$ is constrained to equal B, so we can simplify the expression to $x_j(b) = \sum_i b_{ij}$. This yields the desired Program 6. □

Since the EG program and Program 6 are connected via the same dual, we know that the solution to Program 6 must imply a solution to the primal EG (through computing the implied dual solutions via KKT conditions, which can easily be done) and thus a Lindahl equilibrium. One can also show directly that Program 6 yields Lindahl equilibria. We defer this proof to the section on the capped setting, where we show it for that more general case (Theorem 7).

4.4 A Possible Path to Tâtonnement for Public Goods

Finally, we briefly remark that the dual program (8) can be rewritten in a way that eliminates the variables λ and β_i and thereby turns it into an unconstrained minimization problem. This yields the following program:

$$\min_{\{\lambda_{ij}\}_{ij}} B \cdot \left(\max_{j\in M} \sum_{i\in N} \lambda_{ij}\right) - \sum_{i\in N} B_i \min_{j\in M} \log \frac{\lambda_{ij}}{v_{ij}} \tag{11}$$

One interesting property of this program is that it has a tâtonnement-like interpretation. The λ_{ij} variables can be viewed as personalized prices offered to each agent i for project j. In this interpretation, each agent chooses their favorite projects among those minimizing λ_{ij}/v_{ij}, i.e. ones that maximize their bang-per-buck, and spends their entire budget B_i on such projects. Formally, let $y_i \in \Delta^m$ be such that $y_{ij} > 1$ only when project j minimizes λ_{ij}/v_{ij}. Then $B_i \cdot y_{ij}$ specifies how much of their budget agent i allocates to an optimal bang-per-buck project j. Similarly, let $x \in B \cdot \Delta^m$ be such that $x_j > 0$ only if $\sum_{i\in N} \lambda_{ij} = \max_{j\in M} \sum_{i\in N} \lambda_{ij}$. Then x specifies a budget allocation proposed by the price-setter. Then we have that a subgradient is any $g \in \mathbb{R}^{n\times m}$ such that

$$g_{ij} = x_j - y_{ij}\frac{B_i}{\lambda_{ij}},$$

for any pair (x, y) satisfying the above conditions. This subgradient can be interpreted as a measure of discrepancy. The price-setter is proposing a set of per-agent prices λ and a corresponding budget allocation x. In turn, agent i computes their preferred allocation under λ_i, where $y_{ij}\frac{B_i}{\lambda_{ij}}$ is the amount they would have to spend to obtain $B_i \cdot y_{ij}$ units of project j at price λ_{ij}. The subgradient g_{ij} is then the discrepancy between the price-setter's proposal and the agent's preferred allocation. It is positive (and thus suggests an increase in price) if the agent spends less than the proposed allocation on the project; it is negative (and thus suggests a decrease in price) if the agent spends more. The subgradient is zero exactly when the price-setter's proposed allocation is optimal for each agent, meaning the proposed prices support the allocation. Deriving some form of convergence results for this program would be an interesting direction for future work.

4.5 Proportional Response Dynamics as Mirror Descent

It is known that the Lindahl equilibrium for the uncapped public goods setting can be computed by a simple dynamics [Brandl et al., 2022] which we call the *proportional response* dynamics in analogy to a similar dynamics for private-good Fisher markets [Wu and Zhang, 2007, Zhang, 2011]. At each iteration t, the proportional response dynamics have some current budget allocation $x^t = (x_1^t, \dots, x_j^t)$ summing to B. Let $u_i^t = \langle v_i, x^t \rangle$ be the current utility of agent i under this allocation. Then the next budget allocation in the dynamics is

$$x_j^{t+1} = \sum_{i \in N} B_i \frac{x_j^t v_{ij}}{u_i^t}.$$

This dynamics can be interpreted as each agent i independently deciding how they wish to allocate their share of the budget B_i in the next round. Specifically, agent i allocates spending proportional to how much utility each project provided them at round t. This spending allocation matches the property in (4) we derived earlier from the definition of Lindahl equilibrium. We will show that the proportional response dynamics is the mirror descent algorithm applied to our Program 6.

In order to derive this relationship, we first reformulate Program 6 to an equivalent version: we eliminate the redundant x_j variables, convert the problem into a minimization problem, and define the shorthand function $x_j(b) = \sum_{i \in N} b_{ij}$. Then we get the following convex program:

$$\begin{aligned} \min_{b \geq 0} \quad & f(b) := -\sum_{i \in N, j \in M_i} b_{ij} \left(\log v_{ij} - \log \left(b_{ij} / x_j(b) \right) \right) \\ \text{s.t.} \quad & \sum_{j \in M} b_{ij} = B_i, \ \forall i \in N \end{aligned} \tag{12}$$

Theorem 6. *Assume that the PR dynamics and mirror descent algorithm on Program 12 are both initialized at a point $b^0 \in \mathbb{R}^{n \times m}$ such that $b_i^0 \in B_i \cdot \Delta^m$ and $x_j(b^0) = \sum_{i \in N} b_{ij}^0 > 0$ for all $j \in M$. Then the proportional response dynamics are equivalent to applying the mirror descent algorithm with the entropy reference function to Program 12.*

Proof. Suppose that $x_j(b) > 0$ for all $j \in M$, and thus f is differentiable at b. This holds by assumption for b^0, and we will see that if it holds for the initial point then it holds throughout. The derivative of the objective in Program 12 with respect to b_{ij} is

$$\nabla_{ij} f(b) = -\log v_{ij} + \log b_{ij} + 1 - \log x_j(b) - 1 = -\log v_{ij} + \log(b_{ij}/x_j(b)). \tag{13}$$

In the above, the 1 arises because the derivative of $a \log a$ equals $\log a + 1$ and the -1 arises from the fact that b_{ij} occurs in each of the terms $b_{kj} \log(b_{kj}/x_j(b))$.

Let $x_j^t = \sum_{i \in N} b_{ij}^t$. If we apply the MD update rule in Eq. (1) using the negative entropy reference function and a stepsize $\eta = 1$, we get the update

$$\begin{aligned} b_i^{t+1} &= \arg \min_{b_i \in B_i \cdot \Delta^m} \sum_{j \in M_i} b_{ij} \left(-\log v_{ij} + \log(b_{ij}^t / x_j^t) \right) + \sum_{j \in M_i} b_{ij} \log(b_{ij}/b_{ij}^t) \\ &= \arg \min_{b_i \in B_i \cdot \Delta^m} \sum_{j \in M_i} b_{ij} \left(-\log v_{ij} + \log(b_{ij}/x_j^t) \right) \end{aligned}$$

By Lemma 2(a), the solution to this optimization problem satisfies

$$b_{ij}^{t+1} \propto \exp\left(\log v_{ij} x_j^t \right) \propto v_{ij} x_j^t.$$

In order to get a feasible solution we must normalize the above such that $\sum_{j\in M} b_{ij}^{t+1} = B_i$. Let $u_i^t = \sum_{j\in M} v_{ij}x_j^t$. Applying normalization, we get

$$b_{ij}^{t+1} = \frac{B_i}{u_i^t} v_{ij} x_j^t,$$

Now if we sum over i we get the proportional response dynamics. Moreover, we see that if $x_j^t > 0$ then $x_j^{t+1} > 0$, as long as every public good is valued by at least one agent (public goods valued by nobody can safely be ignored, or preprocessed away). □

Thus, we have shown that the proportional response dynamics is equivalent to mirror descent with unit stepsize. Next we wish to apply the convergence-rate result from Theorem 3. Thus, we need to show that the objective in Program 12 is 1-smooth relative to the entropy reference function.

Lemma 4. *The function f is 1-smooth relative to the reference function $h(b) = \sum_{i\in N, j\in M} b_{ij}\log b_{ij}$, i.e., for all $a, b \in \mathbb{R}^{n\times m}_{>0}$ such that $a_i \in B_i \cdot \Delta^m, b_i \in B_i \cdot \Delta^m$ we have*

$$f(a) \le f(b) + \langle \nabla f(b), a - b\rangle + D_h(a\|b)$$

Proof. Note that f and h are differentiable in the relative interior of $B_i \cdot \Delta^m$. Using (13), we have

$$\begin{aligned}
&f(a) - f(b) - \langle \nabla f(b), a - b\rangle \\
&= -\sum_{i\in N, j\in M_i} a_{ij}(\log v_{ij} - \log \tfrac{a_{ij}}{x_j(a)}) + \sum_{i\in N, j\in M_i} b_{ij}(\log v_{ij} - \log\tfrac{b_{ij}}{x_j(b)}) - \sum_{i\in N, j\in M_i}(a_{ij} - b_{ij})(\log\tfrac{b_{ij}}{x_j(b)} - \log v_{ij}) \\
&= \sum_{i\in N, j\in M_i} a_{ij}\log(a_{ij}/x_j(a)) - \sum_{i\in N, j\in M_i} b_{ij}\log(b_{ij}/x_j(b)) - \sum_{i\in N, j\in M_i}(a_{ij} - b_{ij})\log(b_{ij}/x_j(b)) \\
&= D_g(a\|b),
\end{aligned}$$

where D_g is the Bregman divergence for the entropy-like function $g(b) = \sum_{i\in N, j\in M} b_{ij}\log(b_{ij}/x_j(b))$. It remains to show that $D_g(a\|b) \le D_h(a\|b)$:

$$\begin{aligned}
&D_h(a\|b) - D_g(a\|b) \\
&= \sum_{i\in N, j\in M_i} a_{ij}\log a_{ij} - \sum_{i\in N, j\in M_i} b_{ij}\log b_{ij} - \sum_{i\in N, j\in M_i}(a_{ij} - b_{ij})\log b_{ij} \\
&\quad - \sum_{i\in N, j\in M_i} a_{ij}\log(a_{ij}/x_j(a)) + \sum_{i\in N, j\in M_i} b_{ij}\log(b_{ij}/x_j(b)) + \sum_{i\in N, j\in M_i}(a_{ij} - b_{ij})\log(b_{ij}/x_j(b)) \\
&= \sum_{i\in N, j\in M_i} a_{ij}\log x_j(a) - \sum_{i\in N, j\in M_i} b_{ij}\log x_j(b) - \sum_{i\in N, j\in M_i}(a_{ij} - b_{ij})\log x_j(b) \\
&= \sum_{i\in N, j\in M_i} a_{ij}\log(x_j(a)/x_j(b)) = \sum_{j\in M} x_j(a)\log(x_j(a)/x_j(b)) \ge 0
\end{aligned}$$

The last step follows by noting that the second-to-last expression is the KL divergence between $x(a)$ and $x(b)$, which is always nonnegative. □

Now we can combine Lemma 4 with Theorem 3 to get a $D_h(b^*\|b^0)/t$ rate of convergence for the proportional response dynamics. If we start the dynamics at the uniform allocation $b_{ij}^0 = B_i/m$, we can upper bound the Bregman divergence $D_h(b^*\|b^0)$ as follows:

$$D_h(b^*\|b^0) = h(b^*) - h(b^0) \le -h(b^0) = -\sum_{i\in N, j\in M_i} \tfrac{B_i}{m}\log(B_i/m) = \sum_{i\in N} B_i\log(m/B_i).$$

Combining this with Theorem 3 and Lemma 4, we get a $\frac{\sum_{i \in N} B_i \log(m/B_i)}{t}$ rate of convergence for the proportional response dynamics. Suppose for simplicity that $B = 1$ and $B_i = 1/n$, then we get that proportional response dynamics converges at a rate of $\frac{\log(nm)}{t}$.

The same convergence rate was recently independently obtained by Zhao [2023], after it had been an open question for almost fifty years. Zhao [2023] derived this rate directly, while our result gives a deeper explanation of the performance of the proportional response dynamics: it is equivalent to mirror descent with the entropy reference function applied to Program 6.

5 Capped Public Goods

Next we study the capped public goods setting, where we have a constraint $x_j \leq \text{cap}_j$ for each good $j \in M$. One may naïvely attempt to add this constraint to Program 5 maximizing Nash welfare, but this will not lead to a Lindahl equilibrium and not even to a core solution.

Example 4 (Nash welfare optimum is not a Lindahl equilibrium). *Consider the following instance:*[5]

	B_i	Project 1	Project 2	Project 3	Project 4
Agent 1	2	1	1	0	0
Agent 2	2	1	0	1	0
Agent 3	2	0	0	0	1
cap_j		3	∞	∞	∞

The allocation that maximizes Nash welfare subject to the cap constraints is $x = (3, 0, 0, 3)$. This allocation violates the weak core: consider the blocking coalition $S = \{1, 2\}$ and the objection $z = (3, 0.5, 0.5, 0)$ which gives each $i \in S$ a utility of $u_i(z) = 3.5$ which is strictly higher than $u_i(x) = 3$. Thus, by Proposition 2, x is not a Lindahl equilibrium. This is not an artefact of having zero-valuations; replacing 1s by 10 and 0s by 1 leads to the same situation.

This failure of the Nash rule to extend to capped settings has been noted several times. Suzuki and Vollen [2024, Proposition 4.1] provide an example similar to the one above. Garg et al. [2021, Comment A.1] write that Lindahl equilibrium "does not transform into a Fisher market". While the Nash optimum fails the core, it can be shown that it satisfies a 2-approximation to it [Munagala et al., 2022b, Corollary 3.5].

5.1 Adapting the Convex Program

We will show in this section that Program 6 can be used to compute a Lindahl equilibrium in the capped public goods setting through a simple modification: we simply add a constraint $x_j \leq \text{cap}_j$ for all $j \in M$. Surprisingly, we will show that this works, even though the exact same constraint does not work for the original EG program (Program 5) for maximizing Nash welfare. Thus, we obtain the first efficient algorithm for capped public goods, thereby resolving an open problem first posed by Fain et al. [2016].

[5] This example is similar to a well-known instance in (indivisible) approval-based multi-winner voting where the PAV rule fails the core [Aziz et al., 2017, Peters, 2025, Peters and Skowron, 2020].

Our modified program for capped public goods is as follows, where as before we write $x_j(b) = \sum_{i\in N} b_{ij}$ as a shorthand:

$$\begin{aligned} \max_{b\geq 0} \quad & \sum_{i\in N, j\in M_i} b_{ij} \left(\log v_{ij} - \log(b_{ij}/x_j(b))\right) \\ \text{s.t.} \quad & \sum_{j\in M_i} b_{ij} \leq B_i \text{ for all } i \in N \\ & x_j(b) \leq \text{cap}_j \text{ for all } j \in M \end{aligned} \tag{14}$$

We will require that all valuations have been rescaled such that $v_{ij} > 1$ for all $v_{ij} \neq 0$, to ensure that the coefficients $\log v_{ij}$ in the objective function are positive. Rescaling is without loss of generality, since the Lindahl equilibrium is invariant to scaling valuations by a positive constant. A similar normalization is used by Brandl et al. [2022].

Remark 1 (Need for rescaling.). *Both Program 5 and Program 6 are invariant to rescaling, so why do we need to rescale valuations in the capped setting? The reason is that the spending caps $x_j \leq \text{cap}_j$ may mean that there does not exist a solution satisfying $\sum_{j\in M} b_{ij} = B_i$ for all $i \in N$ (see Example 2). Thus, we modify Program 6 by changing the equality to an inequality. However, when $v_{ij} \leq 1$ and thus $\log v_{ij} \leq 0$ for some i, j, the objective function contains a term for minimizing b_{ij} which may lead to some agents not spending their entire budget under b_{ij} even when it is possible for them to do so. This causes the Lindahl equilibrium correspondence for optimal solutions to fail.*

5.2 The Convex Program Computes a Lindahl Equilibrium

In this section, we will prove that Program 14 computes a zero-respecting Lindahl equilibrium. This in particular proves the existence of such an equilibrium, which does not quite follow from the existence result of Foley [1970], since his model does not allow for caps and does not allow for valuations equal to 0 (since it assumes strictly monotonic valuations).

Our proof proceeds by analyzing KKT conditions (Theorem 1) applied to Program 14. In the notation of Theorem 1, the objective function is obtained by multiplying by -1, giving $f(b) = -\sum_{i\in N, j\in M_i} b_{ij}\log v_{ij} + \sum_{j\in M} h(b_j)$ where b_j is the vector $(b_{ij})_{i\in N_j}$ and h is the function defined as $h(x) = \sum_i x_i \log \frac{x_i}{\sum_k x_k}$ when all x_i are non-negative, and $+\infty$ otherwise.

To apply Theorem 1, we need to compute the subdifferential of f. We first compute the subdifferential of $h(x)$. Note that h is differentiable for all $x \in \mathbb{R}^n_{\geq 0} \setminus \{0\}$, with $\partial h(x) = \{(\log \frac{x_i}{\sum_k x_k})_{i=1}^n\}$. It is not differentiable at $x = 0$, but we can determine its subdifferential using standard calculations.

Lemma 5. *Let $h(x) = \sum_{i=1}^n x_i \log \frac{x_i}{\sum_k x_k}$. Then we have $\partial h(0) = \{g \in \mathbb{R}^n \mid \sum_i e^{g_i} \leq 1\}$.*

Proof. At zero, we have $0\log 0 = 0$, and thus a vector g is a subgradient if and only if

$$h(x) \geq \langle g, x - 0\rangle = \langle g, x\rangle \text{ for all } x \in \mathbb{R}^n. \tag{15}$$

First we note that this is trivially true for $x \notin \mathbb{R}^n_{\geq 0}$, since $h(x) = +\infty$ in that case. It is also true for $x = 0$ since $h(0) = 0$. Thus consider $x \in \mathbb{R}^n_{\geq 0}$ with $x \neq 0$. Let $S = \sum_i x_i$, which is positive. Dividing both sides of (15) by S and writing $p_i = x_i/S$, we see that g is a subgradient if and only if

$$\sum_i p_i \log p_i \geq \langle g, p\rangle \text{ for all } p \in \Delta^n.$$

This in turn can be equivalently written in terms of a minimization problem:

$$\min_{p\in\Delta^n} \langle -g, p\rangle + \sum_i p_i \log p_i \geq 0.$$

By Lemma 2(a), the minimum is attained at $p_j = e^{g_j}/\sum_i e^{g_i}$, giving the value $-\log\left(\sum_i e^{g_i}\right)$ for the left-hand side. This is nonnegative exactly when $\sum_i e^{g_i} \le 1$. □

Since the subdifferential of a sum of convex functions is equal to the sum of the subdifferentials, this allows us to fully characterize the subdifferential of f.

Lemma 6. *Let f be the negative of the objective function of Program 14, and let $b = (b_{ij})_{i\in N, j\in M_i}$ be a feasible point. If there exists some $b_{ij} = 0$ while $x_j(b) > 0$, then $\partial f(b) = \emptyset$. Otherwise, for all vectors $g = (g_{ij})_{i\in N, j\in M_i}$, we have*

$$g \in \partial f(b) \iff \begin{cases} \text{for all } j \in M \text{ s.t. } x_j(b) > 0, \text{ and all } i \in N_j, & g_{ij} = -\log v_{ij} + \log \frac{b_{ij}}{x_j(b)}, \\ \text{for all } j \in M \text{ s.t. } x_j(b) = 0, & \sum_{i\in N_j} e^{g_{ij}+\log v_{ij}} \le 1. \end{cases}$$

Based on this computation of the subdifferential of the objective function of Program 14, we can now prove that an optimal solution to the program will form a Lindahl equilibrium.

Theorem 7. *Assume that valuations are rescaled such that $v_{ij} > 1$ for all $v_{ij} \neq 0$. Let x^* be an optimal solution to Program 14. Then there exist zero-respecting prices p such that (x^*, p) forms a Lindahl equilibrium for the capped public goods setting.*

Proof. We apply the KKT conditions of Theorem 1 to Program 14. For convenience, let us label the constraints of the program using the notation of the KKT conditions:

$$\begin{aligned} h_i^{(1)}(b) &= \textstyle\sum_{j\in M} b_{ij} - B_i && \text{for all } i \in N, \\ h_j^{(2)}(b) &= \textstyle\sum_{i\in N} b_{ij} - \text{cap}_j && \text{for all } j \in M, \\ h_{ij}^{(3)}(b) &= -b_{ij} && \text{for all } i \in N \text{ and } j \in M_i. \end{aligned}$$

All these functions are affine and thus differentiable, with singleton subdifferentials.

Let b^* be an optimal solution to Program 14. By Theorem 1, we know that there exists a subgradient $g^* \in \partial f(b^*)$ such that

$$g^* + \sum_{i\in N} \lambda_i \nabla h_i^{(1)}(b^*) + \sum_{j\in M} \mu_j \nabla h_i^{(2)}(b^*) + \sum_{i\in N, j\in M_i} \eta_{ij} \nabla h_{ij}^{(3)}(b^*) = 0, \tag{16}$$

with $\lambda_i, \mu_j, \eta_{ij} \ge 0$ and such that complementary slackness holds. Equation (16) can be equivalently states as

$$g_{ij}^* + \lambda_i + \mu_j - \eta_{ij} = 0 \quad \text{for all } i \in N \text{ and } j \in M. \tag{17}$$

Let us now understand the implications of (16). We go through each project $j \in M$, and distinguish the cases where $x_j^* = \text{cap}_j$, where $0 < x_j^* < \text{cap}_j$, and where $x_j^* = 0$.

- Consider a project $j \in M$ with $x_j^* = \text{cap}_j$. Let $i \in N_j$. Note that $b_{ij}^* > 0$, since otherwise the subdifferential is empty, contradicting $g^* \in \partial f(b^*)$. Thus, by complementary slackness, $\eta_{ij} = 0$. Then the ij'th-component of (16) implies

  $$0 = -\log v_{ij} + \log \frac{b_{ij}^*}{x_j^*} + \lambda_i + \mu_j$$

  and thus, because $\mu_j \ge 0$, that

  $$0 \ge -\log v_{ij} + \log \frac{b_{ij}^*}{x_j^*} + \lambda_i.$$

  Rearranging and exponentiating both sides, we conclude that

  $$v_{ij} \ge e^{\lambda_i} \frac{b_{ij}^*}{x_j^*}.$$

- Consider a project $j \in M$ with $0 < x_j^* < \text{cap}_j$. Let $i \in N_j$. Again note that $b_{ij}^* > 0$ since otherwise the subdifferential $\partial f(b^*)$ is empty. Then, by complementary slackness, $\mu_j = \eta_{ij} = 0$. Thus, the ij'th-component of (16) implies

$$0 = g_{ij}^* + \lambda_i = -\log v_{ij} + \log \frac{b_{ij}^*}{x_j^*} + \lambda_i.$$

  Rearranging and exponentiating both sides, we conclude that

$$v_{ij} = e^{\lambda_i} \frac{b_{ij}^*}{x_j^*}.$$

- Finally, consider a project $j \in M$ with $x_j^* = 0$. Thus by complementary slackness, $\mu_j = 0$. Let $i \in N_j$. By definition of x_j^*, we have $b_{ij} = 0$. The ij'th-component of (16) implies

$$0 = g_{ij}^* + \lambda_i - \eta_{ij}.$$

  Writing $w_{ij} = g_{ij}^* + \log v_{ij}$ and noting that $\eta_{ij} \geq 0$, we get

$$0 \leq w_{ij} - \log v_{ij} + \lambda_i.$$

  Rearranging and exponentiating both sides, we conclude that

$$v_{ij} \leq e^{\lambda_i} e^{w_{ij}}.$$

Collecting all our conclusions, we have found that for $i \in N$ and $j \in M_i$,

$$v_{ij} \begin{cases} \geq e^{\lambda_i} \frac{b_{ij}^*}{x_j^*} & \text{if } x_j^* = \text{cap}_j, \\ = e^{\lambda_i} \frac{b_{ij}^*}{x_j^*} & \text{if } 0 < x_j^* < \text{cap}_j, \\ \leq e^{\lambda_i} e^{w_{ij}} & \text{if } x_j^* = 0. \end{cases} \tag{18}$$

We now form a Lindahl equilibrium using the following prices for $i \in N$ and $j \in M_i$:

$$p_{ij} = \begin{cases} \frac{b_{ij}^*}{x_j^*} & \text{if } x_j^* > 0, \\ e^{w_{ij}} & \text{if } x_j^* = 0. \end{cases} \tag{19}$$

For i, j such that $v_{ij} = 0$, we set $p_{ij} = 0$ and $b_{ij}^* = 0$. It follows from these definitions that the prices are zero-respecting. Note that with these prices, the identity $p_{ij} x_j^* = b_{ij}^*$ holds for all $i \in N$ and $j \in M$ (by case analysis on whether $x_j^* = 0$).

We claim that (x^*, p) forms a Lindahl equilibrium.

For profit maximization, note that if $x_j^* > 0$, then $\sum_{i \in N} p_{ij} = \sum_{i \in N_j} b_{ij}^*/x_j^* = 1$ by definition of x_j^*, and if $x_j^* = 0$, then $\sum_{i \in N} p_{ij} = \sum_{i \in N_j} e^{w_{ij}} \leq 1$ from the subdifferential characterization in Lemma 6.

For the affordability condition, for each $i \in N$ we have

$$\langle p_i, x^* \rangle = \sum_{j \in M} p_{ij} x_j^* = \sum_{j \in M} b_{ij}^* \leq B_i,$$

using the identity $p_{ij} x_j^* = b_{ij}^*$ and the feasibility of b^* in Program 14.

It remains to prove utility maximization. Fix an agent $i \in N$. We will show that x^* is utility maximizing subject to the budget constraint $\langle p_i, x \rangle \leq B_i$. We divide the proof of this into two parts, based on whether agent i spends their entire budget under p_i or not.

First suppose that $\sum_{j \in M} b_{ij}^* < B_i$. We want to show that this only occurs when all projects $j \in M$ with $v_{ij} > 0$ have $x_j^* = \text{cap}_j$. Suppose for a contradiction that $x_j^* < \text{cap}_j$ for some $j \in M_i$. By complementary slackness, we have $\lambda_i = \mu_j = 0$. Thus, (16) implies $g_{ij}^* \geq 0$. If $x_j^* > 0$, then we have $g_{ij}^* = -\log v_{ij} + \log(b_{ij}/x_j)$ and thus $g_{ij}^* < 0$ because $v_{ij} > 1$, a contradiction. Otherwise, if $x_j^* = 0$, then $e^{g_{ij}^* + \log v_{ij}} \le 1$ by the subdifferential characterization in Lemma 6, which implies $g_{ij}^* + \log v_{ij} \le 0$. Hence $g_{ij}^* \le -\log v_{ij} < 0$ since $v_{ij} > 1$, again a contradiction.

Thus we have shown that if agent i does not spend their whole budget, then they are already achieving the maximal possible utility under *any* feasible allocation (because $x_j = \text{cap}_j$ for all $j \in M$ such that $v_{ij} > 0$), and thus x^* is utility maximizing for i.

Next consider the case where $\sum_{j\in M} b_{ij}^* = B_i$. Combining (18) and (19), we have that the "bang per buck" of project $j \in M_i$ satisfies

$$\frac{v_{ij}}{p_{ij}} \begin{cases} \ge e^{\lambda_i} & \text{if } x_j^* = \text{cap}_j, \\ = e^{\lambda_i} & \text{if } 0 < x_j^* < \text{cap}_j, \\ \le e^{\lambda_i} & \text{if } x_j^* = 0. \end{cases} \tag{20}$$

Now, an affordable bundle y is utility maximizing for i (among bundles satisfying the cap constraints) if and only if (i) for every project with $\frac{v_{ij}}{p_{ij}} > e^{\lambda_i}$, we have $y_j = \text{cap}_j$, and (ii) for every project with $\frac{v_{ij}}{p_{ij}} < e^{\lambda_i}$, we have $y_j = 0$, and (iii) the whole budget is spent. Because x^* is such a bundle, it is utility maximizing for i.

More formally, let $y = (y_j)_{j\in M}$ be an allocation such that $0 \le y_j \le \text{cap}_j$ for all $j \in M$ and $\langle p_i, y\rangle \le B_i$. Then for every $j \in M$ with $x_j^* = 0$ we have $y_j - x_j^* \ge 0$, and for every $j \in M$ with $x_j* = \text{cap}_j$ we have $y_j - x_j^* \le 0$. Thus

$$\begin{aligned} u_i(y) - u_i(x^*) &= \sum_{j\in M} v_{ij}(y_j - x_j^*) \\ &= \sum_{\substack{j\in M_i \\ x_j^*=0}} v_{ij}(y_j - x_j^*) + \sum_{\substack{j\in M_i \\ 0<x_j^*<\text{cap}_j}} v_{ij}(y_j - x_j^*) + \sum_{\substack{j\in M_i \\ x_j^*=\text{cap}_j}} v_{ij}(y_j - x_j^*) \\ &\overset{(20)}{\le} \sum_{\substack{j\in M_i \\ x_j^*=0}} e^{\lambda_i} p_{ij}(y_j - x_j^*) + \sum_{\substack{j\in M_i \\ 0<x_j^*<\text{cap}_j}} e^{\lambda_i} p_{ij}(y_j - x_j^*) + \sum_{\substack{j\in M_i \\ x_j^*=\text{cap}_j}} e^{\lambda_i} p_{ij}(y_j - x_j^*) \\ &= e^{\lambda_i} \sum_{j\in M_i} p_{ij}(y_j - x_j^*) = e^{\lambda_i}(\langle p_i, y\rangle - \langle p_i, x^*\rangle) \le 0. \end{aligned}$$

In the last line we used that (x, p) is zero-respecting for the second equality, and we used that $\langle p_i, y\rangle \le B_i = \langle p_i, x^*\rangle$ for the last inequality. It follows that $u_i(y) \le u_i(x^*)$, establishing the utility maximization condition. □

5.3 Discussion of the Convex Program

Comparison to Fisher markets. It is interesting to contrast our program with the Fisher market setting with private goods. There, the Eisenberg–Gale program also does not allow the introduction of saturating constraints on the primal variables (which correspond to a maximum amount of a good that an agent may receive). Yet it is not possible to add such constraints to the Shmyrev program for Fisher markets either, because that program does not contain the original primal variables encoding the allocation (in contrast to our public-goods program). Instead, the allocation is obtained through a nonlinear function of the optimization variables in the Shmyrev program.[6] Thus, Program 6 allows for a type of saturating consumption constraint that has previously never been possible for either private or public goods.

[6] It is known that *spending constraints* on a per-buyer basis can be introduced to the Shmyrev program [Birnbaum et al., 2011], but these are very different from saturating constraints on the primal variables.

Discontinuity as $v_{ij} \to 0$. Given that our program computes a Lindahl equilibrium that is zero-respecting, its output is not continuous as $v_{ij} \to 0$. This is unavoidable due to Example 2 (see Example 2), and unsurprising in light of our normalization of valuations.

Not all Lindahl equilibria are optimal solutions. In the uncapped setting, *every* Lindahl equilibrium forms an optimum of both Program 6 and the Eisenberg–Gale program. As the following example shows, this is not the case for the capped setting, where Program 14 captures only a strict subset of Lindahl equilibria. The example also shows that Lindahl equilibria are not unique in utilities.

Example 5 (Lindahl equilibrium is not unique in utilities). *Consider the following instance:*

	B_i	Project 1	Project 2	Project 3
Agent 1	1	1	1	0
Agent 2	1	1	0	1
cap_j		1	∞	∞

This instance is cap-sufficient, since each agent has a positive valuation for an uncapped project. Let us determine the set of zero-respecting Lindahl equilibria (x, p). By Corollary 1, x is Pareto-optimal, and therefore $x_1 = 1$ and $x_2 + x_3 = 1$. For each $\gamma \in [0, 1]$, one can check that $x = (1, 1 - \gamma, \gamma)$ forms a Lindahl equilibrium together with the prices $p_1 = (\gamma, 1, 0)$ and $p_2 = (1 - \gamma, 0, 1)$. It follows that, in the capped setting, Lindahl equilibria are not unique in utilities: in the equilibrium allocation $(1, 1, 0)$, agent 1 obtains utility 2, but in the equilibrium allocation $(1, 0, 1)$, agent 1 obtains utility 1.

Note that the allocation $x^* = (1, \frac{1}{2}, \frac{1}{2})$ is the unique allocation that is intuitively fair and respects the symmetry of the instance, but this allocation is not the only Lindahl equilibrium. However, Program 14 uniquely selects x^*, because on this instance its objective function simplifies to $-b_{11} \log b_{11} - b_{21} \log b_{21}$ which is maximized by $b_{11} = b_{21} = 0.5$, leaving each agent a budget of 0.5 to spend on other projects.

An intuitive reason for why our program does not capture all Lindahl equilibria is that the KKT conditions that we analyzed in the proof of Theorem 7 impose an additional constraint on the contributions of agents to projects that are fully funded ($x_j = \text{cap}_j$), saying that every agent's bang-per-buck ratio for that good should exceed their "normal" bang-per-buck ratio e^{λ_i} by a common agent-independent factor e^{μ_j}. On the above example, this leads the program to select the most natural equilibrium (and this remains the case if the caps and endowments are varied), suggesting that our convex program program might define a desirable decision rule for selecting Lindahl equilibria.

5.4 Computation and Experiments

Let us briefly discuss how to solve Program 14. Numerically, the program can be solved using any conic convex optimization solver supporting exponential cones, such as MOSEK, COPT, Clarabel, ECOS, or SCS, by formulating the program as

$$\begin{aligned} \max_{b \geq 0, x \geq 0, t} \quad & \textstyle\sum_{i \in N, j \in M_i} b_{ij} \log v_{ij} - t_{ij} \\ \text{s.t.} \quad & (x_j, b_{ij}, -t_{ij}) \in \mathcal{K}_{\exp} \text{ for all } i \in N, j \in M_i \\ & \textstyle\sum_{j \in M_i} b_{ij} \leq B_i \text{ for all } i \in N \\ & x_j = \textstyle\sum_{i \in N_j} b_{ij} \text{ for all } j \in M \\ & x_j \leq \text{cap}_j \text{ for all } j \in M \end{aligned}$$

where $\mathcal{K}_{\exp} = \{(x_1, x_2, x_3) : x_1 \geq x_2 e^{x_3/x_2}\}$ is the (primal) exponential cone. We built a simple online tool for solving moderate-size instances with the SCS solver [O'Donoghue et al., 2016], available at

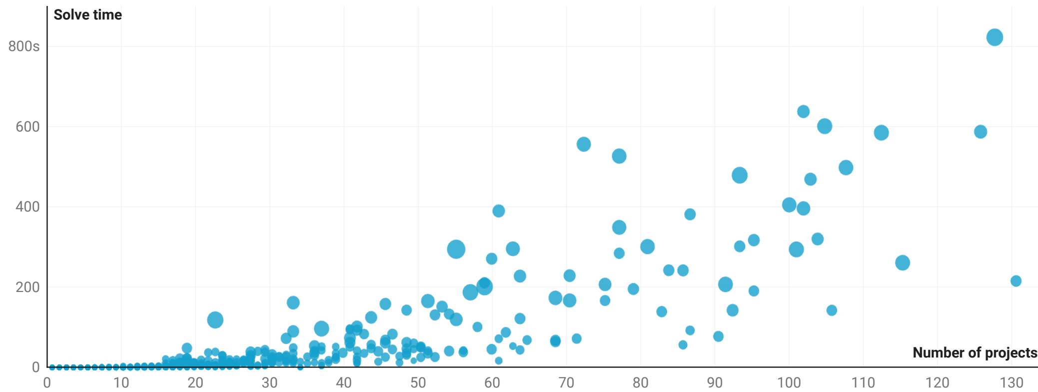

Fig. 2. Results of our experiments on Pabulib instances, showing the solve time of the MOSEK solver as a function of the number of projects in the instance. The largest instances are from Warsaw and Amsterdam.

dominik-peters.de/demos/lindahl/. From a complexity-theoretic perspective, an ε-optimal solution to Program 14 can be computed in polynomial time using the ellipsoid method [see, e.g., Vishnoi, 2021, Theorem 13.1].

To evaluate the performance of computing Lindahl equilibrium via Program 14, we implemented it using the MOSEK solver and applied it to the participatory budgeting datasets in the Pabulib repository [Faliszewski et al., 2023]. We find that the program can be solved quite quickly, with solve times shown in Section 5.4. The longest solve time we encountered was 822s (or 1489s including the time to write down the encoding) for an instance from Warsaw with 14 897 voters (with 11 426 distinct approval sets) and 134 projects.

For the uncapped setting, Zhao and Freund [2023, Section 4.2] present some experiments on the performance of the proportional response dynamics, and find that it outperforms several alternative solution methods.

5.5 Computing Core Allocations for Separable Piecewise-Linear Concave Utilities

We have set up the capped setting with the caps interpreted as an exogenous constraint. An alternative interpretation is as a capped utility function $u_i(x) = \sum_{j \in M} v_{ij} \min(x_j, \mathrm{cap}_j)$. This view suggests a variety of generalizations: for example, we might want to allow different agents to specify different caps. We can generalize further to *separable piecewise-linear concave utilities* (SPLC). These are utility functions that can be written as a sum over goods (separable), with the term corresponding to a good being a (non-decreasing) piecewise-linear concave function of x_j. See Figure 3 for an example.

This class of utility functions is well-studied for private goods, both for Fisher markets and Arrow–Debreu exchange markets. For these markets, just as for linear utilities, equilibrium exists and is rational under mild conditions; however computing an equilibrium becomes PPAD-complete [Chen and Teng, 2009, Deligkas et al., 2024, Vazirani and Yannakakis, 2011]. A complementary pivot algorithm for computing an equilibrium has been proposed [Garg et al., 2015]. This algorithm is based on linear complementarity [Eaves, 1971, 1976], which interestingly can also be used to show existence of Lindahl equilibria [Munagala et al., 2022b, Appendix A].

We leave the problem of computing Lindahl equilibria for SPLC utilities open, but we show how our result for the capped setting can be used to at least compute a core-stable allocation (up to any desired approximation factor). We do this by reducing an SPLC instance to a capped instance

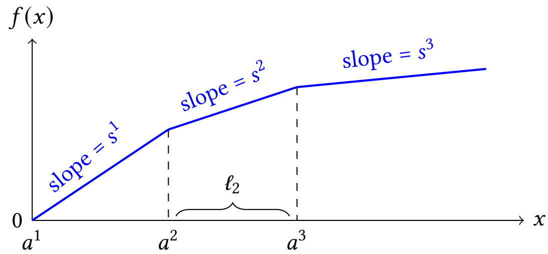

Fig. 3. An example of a piecewise-linear concave function.

(with each piece of the piecewise-linear utilities becoming its own separate good), and show that a Lindahl equilibrium for this instance is core-stable with respect to the SPLC utilities.

We begin with formal definitions. A function $f : [0, B] \to \mathbb{R}$ with $f(0) = 0$ is *piecewise-linear concave* if it can be decomposed into a finite number of linear *segments*, specified by their lengths $\ell^1, \dots, \ell^k$ and slopes $s^1, \dots, s^k$ with $s^1 \geq \dots \geq s^k \geq 0$. Explicitly, for each $i \in [k]$, writing $a^i = \ell^1 + \dots + \ell^{i-1}$ for the left endpoint of the ith segment, we have $f(a^i + x) = f(a^i) + s^i x$ for $x \in [0, \ell^i]$. Figure 3 shows an example. A utility function u_i is called *separable piecewise-linear concave* (SPLC) if there exist piecewise-linear concave functions f_{ij} for all $j \in M$ such that $u_i(x) = \sum_{j \in M} f_{ij}(x)$ for all allocations x. Note that by subdividing segments if necessary, we may assume that for each project j, all agents i agree on the total number k_j of segments in f_{ij} as well as their lengths. At the same time, we can choose this common subdivision in a minimal way, so that from one segment to the next, there is always at least one agent whose slope *strictly* decreases. Let us write s_{ij}^t for the slope of the tth segment of f_{ij}, and let us write ℓ_j^t for the length of the tth segment for project j.

Any instance of the public goods problem with SPLC utility functions can be translated into an instance with linear utility functions and with caps, using the following construction.

Definition 6. *Suppose we are given an SPLC instance I specified by the slopes $(s_{ij}^t)_{i \in N, j \in M, t \in [k_j]}$ and lengths $(\ell_j^t)_{j \in M, t \in [k_j]}$ of the segments. We construct a public goods instance I' with linear utility functions on the same set of agents N and the new set of projects $M' = \bigcup_{j \in M} \{j^1, \dots, j^{k_j}\}$ with $\mathrm{cap}_{j^t} = \ell_j^t$. The project j^t will describe how much of project j will be funded in the area of its tth segment. Finally, we take valuations $v_{ij^t} = s_{ij}^t$.*

Let us say that an SPLC instance is *well-behaved* if the derived instance is cap-sufficient in the sense of Definition 3. This is guaranteed to be the case, for example, if for every agent i, the total length of segments with positive slope across all projects is at least B. Similar sufficient conditions are used for private goods equilibria [e.g., Vazirani and Yannakakis, 2011, Section 2].

Theorem 8. *Let I be a well-behaved instance of the public goods problem with SPLC utilities. Then any zero-respecting Lindahl equilibrium for the instance I' as constructed in Definition 6 can be transformed into a core-stable allocation for the SPLC instance I.*

Proof. Suppose that x' is a zero-respecting Lindahl equilibrium allocation for I'. By Corollary 1, x' is Pareto-optimal. Note that we always have $s_{ij^t} \geq s_{ij^{t-1}}$ by concavity, and for at least one agent the inequality is strict (by minimality of the chosen common subdivision). Thus, Pareto-optimality implies that if $x'_{j^t} > 0$ then $x'_{j^{t-1}} = \mathrm{cap}_{j^t}$ (which equals ℓ_j^t). This allows us to define an allocation x for instance I by setting $x_j = \sum_{t=1}^{k_j} x'_{j^t}$. We now argue that x is core stable.

Suppose not, and there is some blocking coalition $S \subseteq N$ and objection $z = (z_j)_{j \in M}$ such that $\sum_{j \in M} z_j \leq \sum_{i \in S} B_i$ and for all $i \in S$, we have $u_i(z) \geq u_i(x)$, with strict inequality for at least $i \in S$.

We construct a core objection z' for x', contradicting its core stability (Proposition 2). For each $j \in M$, let t be chosen minimal such that $\ell_j^1 + \cdots + \ell_j^{t-1} \geq z_j < \ell_j^1 + \cdots + \ell_j^t$. Then set $z'_{jr} = \text{cap}_{jr} = \ell_j^r$ for $r = 1, \ldots, t-1$ and $z'_{jt} = z_j - (\ell_j^1 + \cdots + \ell_j^{t-1})$, as well as $z'_{jr} = 0$ for $r = t+1, \ldots, k_j$. Then it is easy to see that $u_i(z') = u_i(z)$ for all i, and similarly $u_i(x') = u_i(x)$ for all i, and thus z' is a core deviation to x', a contradiction. □

6 Conclusion

We have developed a new class of convex programs that can be used to efficiently compute Lindahl equilibria both in the uncapped and the capped setting. These new programs open up many opportunities for future research.

In the uncapped setting, our new program might lead to new proofs of known results for the well-studied maximum Nash welfare rule. This might include the result about participation incentives of Brandl et al. [2022] or the axiomatic characterization of Guerdjikova and Nehring [2014]. Perhaps our program could also shed light on the other uses of the Eisenberg–Gale program across statistics, information theory, and medical imaging, as discussed in Section 1.2. For the capped setting, our computability result has implications for the discrete public goods model, because it allows the efficient implementation of the 9.27-approximation to the core obtained by Munagala et al. [2022b], rather than having to rely on their 67.37-approximation. Munagala et al. [2022b] used Lindahl equilibrium as a black box to obtain their approximation result; reasoning about the structure of our convex program program might lead to even better bounds.

Since our focus has been on computational questions, we have not considered strategic aspects. Lindahl equilibrium is well-known to have high informational requirements, and in particular we need to know the truthful valuations of the agents to compute it. Interpreted as a decision rule [Gul and Pesendorfer, 2020], Lindahl equilibrium is not strategyproof and can be manipulated both in a free-riding sense [Brandl et al., 2021, Section 5.3], and in some paradoxical ways [Aziz et al., 2020, Theorem 3(ii)], even in the uncapped setting. Manipulability is unavoidable if one desires a Pareto-efficient and core-stable solution, both in the uncapped setting [Brandl et al., 2021, Theorem 2 and Theorem 3] and in the capped setting [Bei et al., 2024, Theorem 6.2]. These impossibilities apply even for approval (0/1) preferences. For more general linear utilities, strategyproofness is only attainable by dictatorial-type rules [Hylland, 1980], even in the uncapped setting.

We leave several interesting technical questions open. Is the optimum of our program unique in utilities? This is known to be true for the uncapped setting, by strict convexity (in utilities) of the Eisenberg–Gale program. Can we develop first-order methods for the capped settings, or derive a natural dynamics converging to an equilibrium? Applying mirror descent to our program does not appear to lead to a nice closed-form update like in the uncapped setting. Finally, can the cap constraint be generalized? For example, one could apply cap constraints on the total spending of *sets* of public goods. This would allow us to model multi-issue and multi-round decision making settings [see, e.g., Banerjee et al., 2023, Section 5]. It would also allow us to embed private goods in the model [as in Conitzer et al., 2017], and potentially connect the notions of Fisher market equilibrium and Lindahl equilibrium.

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