Structural Tractability in Hedonic Games

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Efficient Coalition Formation when Agents' Preferences are Structured:

Find stable outcomes	Maximise social welfare	Allocate goods fairly	Faster algorithms for games that matter

Hedonic Games: The Model

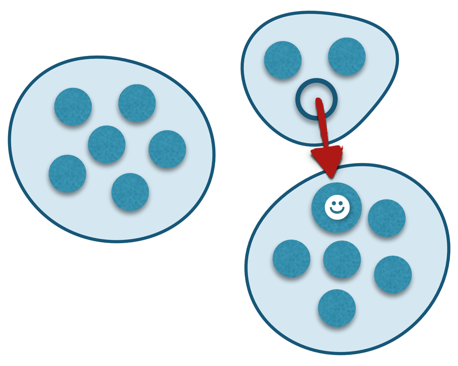
Finite set N of agents, each $i \in N$ having preferences $\succcurlyeq_i$ over groups of agents:

$$\{1,2\} \succ_1 \{1,2,3\} \succ_1 \{1\} \succ_1 \{1,3\}$$

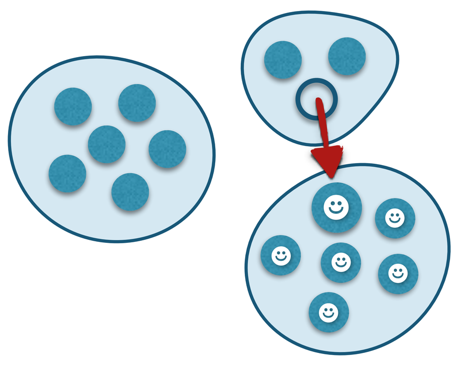
Outcome: a partition π of the agent set N.

Solution Concepts

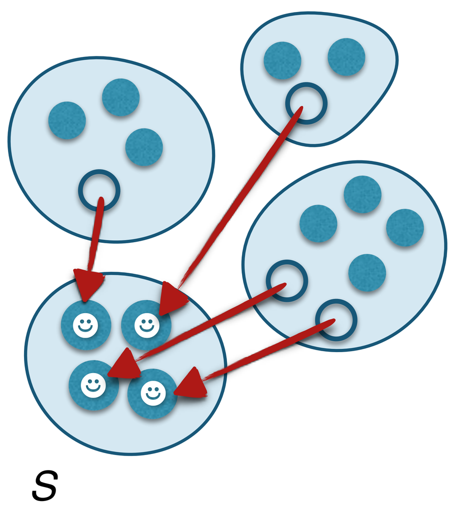
We want partition π to be *stable*. What could this mean?



Nash stable (NS): no individual wants to change into another group.



Individually stable (IS): no individual can change into a group in which all members welcome her.



Core stable (CR): no group S of agents all prefer S to where they are in π.

Strict Core stable (SCR): only require 1 deviator in S to have a strict preference.

Social Welfare: if agents have numerical *utilities*, we can aim to find a partition that has high social welfare, i.e., the sum of the agents' utilities is large.

Envy-free: We can also define *fairness* concepts: a partition is envy-free if no agent wishes to take the position ofo another agent.

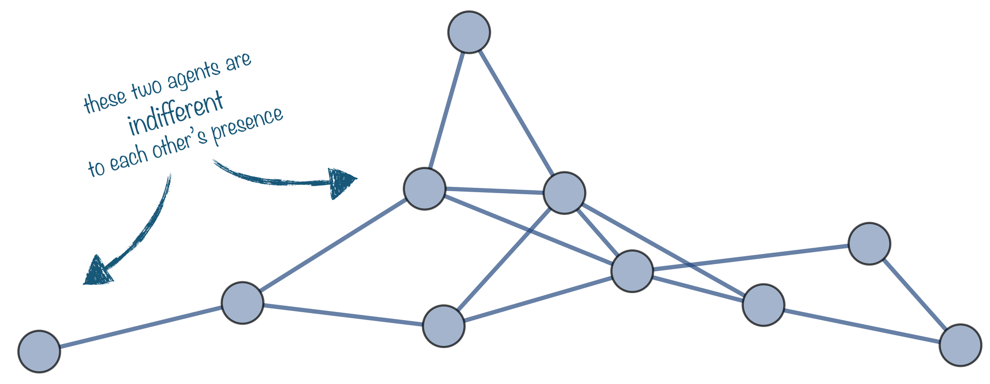
Computational Complexity: Bad News

Given: a hedonic game (concisely represented somehow)
Question: Does there exist a *good* outcome in one of the senses above? If yes, find one.

- Large AI literature studies complexity of this question
- Almost always **intractable**: questions are hard for NP or even for the second level of the polynomial hierarchy
- Previous attempts at identifying islands of tractability has focussed on restricting individual preferences, say by considering preferences obtained by adding, averaging, or minimising values.
- Peters and Elkind (IJCAI 2015): this approach can't work
- New approach needed: let's try a structural one

Graphical Hedonic Games

Idea: Agents are part of a **social network**, and we try to exploit the structure of the network topology.



Formally, we equip a hedonic game with an **agent dependency graph** $G = (N, E)$ so that agents only care about whether they are together with their neighbours:

$$S \succcurlyeq_i T \iff S \cap \mathrm{Ngbs}_G(i) \succcurlyeq_i T \cap \mathrm{Ngbs}_G(i)$$

Note: Can always take the complete graph, but we usually want few edges.

Structure in the Network Topology

- If the social network of a graphical hedonic game is structured in some way, algorithms may be able to use this structure.
- Throughout algorithmic graph theory, bounding treewidth is a wildly successful technique for obtaining tractability.
- We report results for bounded treewidth and bounded max-degree below.
- Future work: Other structural restrictions: planar graphs? minor-free graphs? bipartite graphs?

Results: Bounded Treewidth and Degree

Theorem. Deciding whether a graphical hedonic game admits a stable outcome is linear-FPT w.r.t. the treewidth k and max-degree d of the underlying graph; that is, the problem can be solved in time $O(f(k,d) \cdot n)$ where f is a computable function.

Proof method: Define "HG-logic" to capture properties of hedonic games, translate to MSO, apply Courcelle's theorem.

Theorem. Maximising utilitarian or egalitarian social welfare in a graphical hedonic game can be done in time $\tilde{O}(2^{kd^2}n)$. We can also find a Nash- or individually stable outcome in time $\tilde{O}(2^{kd^5}n)$.

Theorem. It is necessary to bound the max-degree: finding a stable outcome remains NP-hard for graphical hedonic games of treewidth 2, but of unbounded degree.

Application: Allocation of Indivisible Goods

- A collection of objects need to be allocated to agents
- Agents have preferences over bundles of objects.
- Goal: a fair allocation, or a welfare-maximising allocation (= combinatorial auctions).
- This model can be seen as a special case of hedonic games, where objects are agents that are indifferent between all outcomes.
- Positive results above transfer to this setting.

Presented at AAAI 2016, Phoenix, Arizona