Optimal Bounds for the No-Show Paradox via SAT Solving

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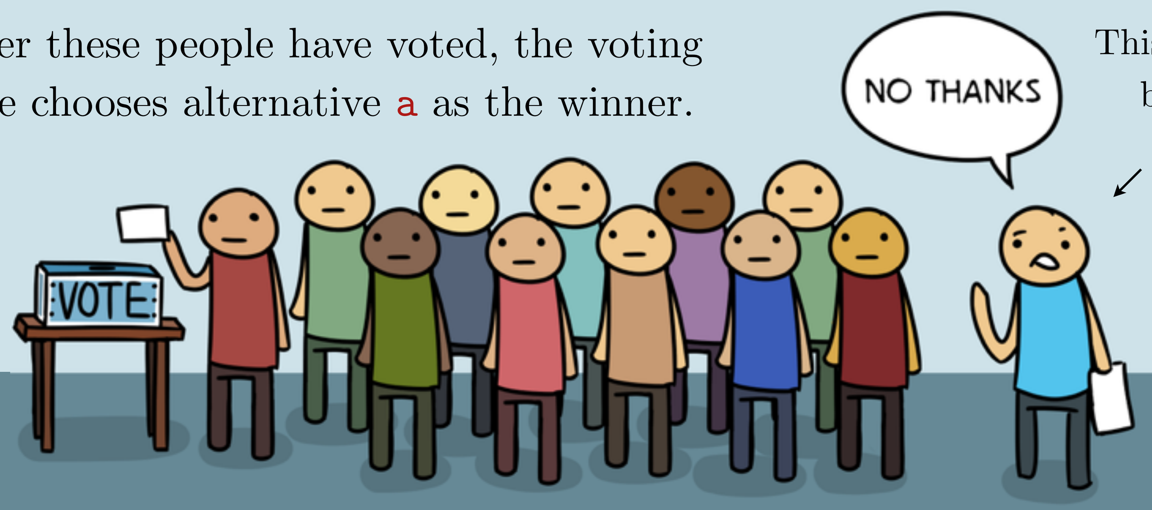
After these people have voted, the voting rule chooses alternative a as the winner.



This guy was about to submit his truthful ballot abcd: but even though a is his most-preferred outcome, the voting rule would suddenly choose b as the winner if he were to submit his vote.

So he is better off staying at home.

Moulin's Theorem (1988):
Every Condorcet-consistent voting rule fails participation when there are at least 25 voters and 4 alternatives.

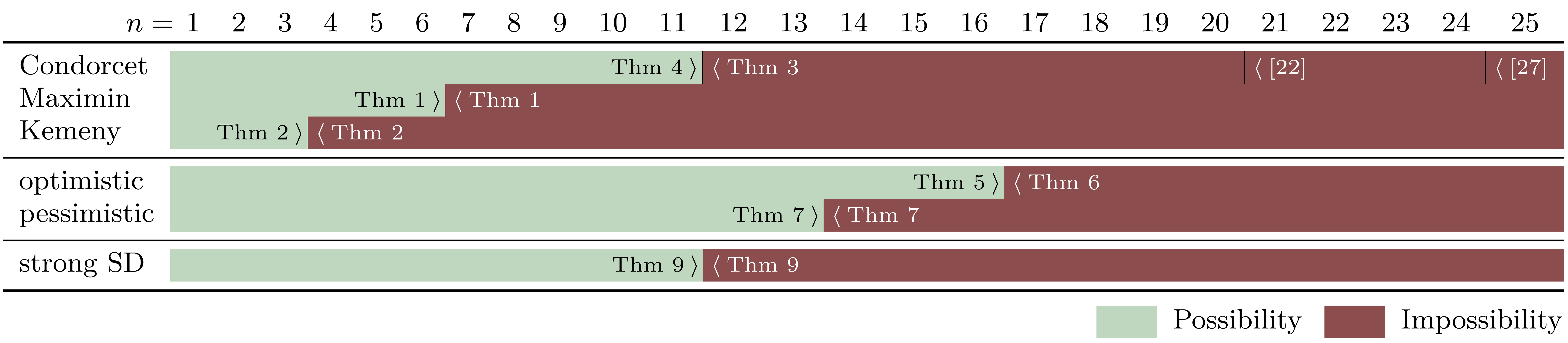
- ↑ the voting rule must choose the "obvious" winner if one exists: alternative a is a *Condorcet winner* if it wins by a majority against every other alternative in a pairwise comparison
- ↑ the "no-show paradox": there is a situation (i.e., a preference profile) where a voter is better off abstaining from the election rather than voting truthfully.
- ↑ our *key question*: is this tight? does this paradox occur only with this many voters, or does it occur even with fewer voters? can we avoid it if there are not too many voters?

We use **computer-assisted proof techniques**:
- Restrict to **finite instance** (say 12 voters, 4 alternatives)
- Encode axioms as clauses in a CNF formula
- Use a SAT solver: satisfiable → good voting rule; unsatisfiable → impossibility result
- Use MUS extraction to find a **human-readable** proof

This paper: extremely large formulas (100m+ variables)
We use "**incremental proof discovery**" by iteratively proving stronger results while using knowledge from proofs generated for weaker results.

For which number of voters can we avoid the no-show paradox?

Tight bounds for resolute, set-valued, and probabilistic voting rules:



For up to **11 voters** and **4 alternatives**, there exists a Condorcet-consistent voting rule that satisfies participation

This Condorcet extension is found by the SAT solver, and given by a lookup table like below. The voting rule found is also pairwise, Pareto-optimal, a refinement of the top cycle, and picks a maximin winner in 99.8% of cases.

```
a,#1,(1,1,1,1,1,1)          a,#11,(9,11,3,9,1,-9)
a,#1,(1,1,1,1,1,-1)         a,#11,(11,9,3,7,1,-9)
a,#1,(1,1,1,-1,1,1)         c,#11,(5,-9,-1,-11,-1,7)
a,#1,(1,1,1,-1,-1,1)        c,#11,(5,-9,-1,-11,-1,5)
a,#1,(1,1,1,1,-1,-1)        c,#11,(3,-11,-1,-9,1,7)
a,#1,(1,1,1,-1,-1,-1)       c,#11,(3,-11,-3,-9,1,7)
b,#1,(-1,1,1,1,1,1)         c,#11,(3,-11,-3,-11,-1,7)
b,#1,(-1,1,1,1,1,-1)        b,#11,(-1,3,-5,-3,5,-3)
b,#1,(-1,-1,1,1,1,1)        b,#11,(-3,3,-7,-3,5,-3)
b,#1,(-1,-1,-1,1,1,1)       b,#11,(-3,1,-7,-3,5,-3)
b,#1,(-1,1,-1,1,1,-1)       c,#11,(-3,1,-5,-5,5,-1)
b,#1,(-1,-1,-1,1,1,-1)      a,#11,(3,7,11,-3,9,11)
c,#1,(1,-1,1,-1,1,1)        a,#11,(3,7,11,-3,9,9)
c,#1,(1,-1,1,-1,-1,1)       a,#11,(3,7,11,-5,9,11)
```

Further results: set-valued, probabilistic

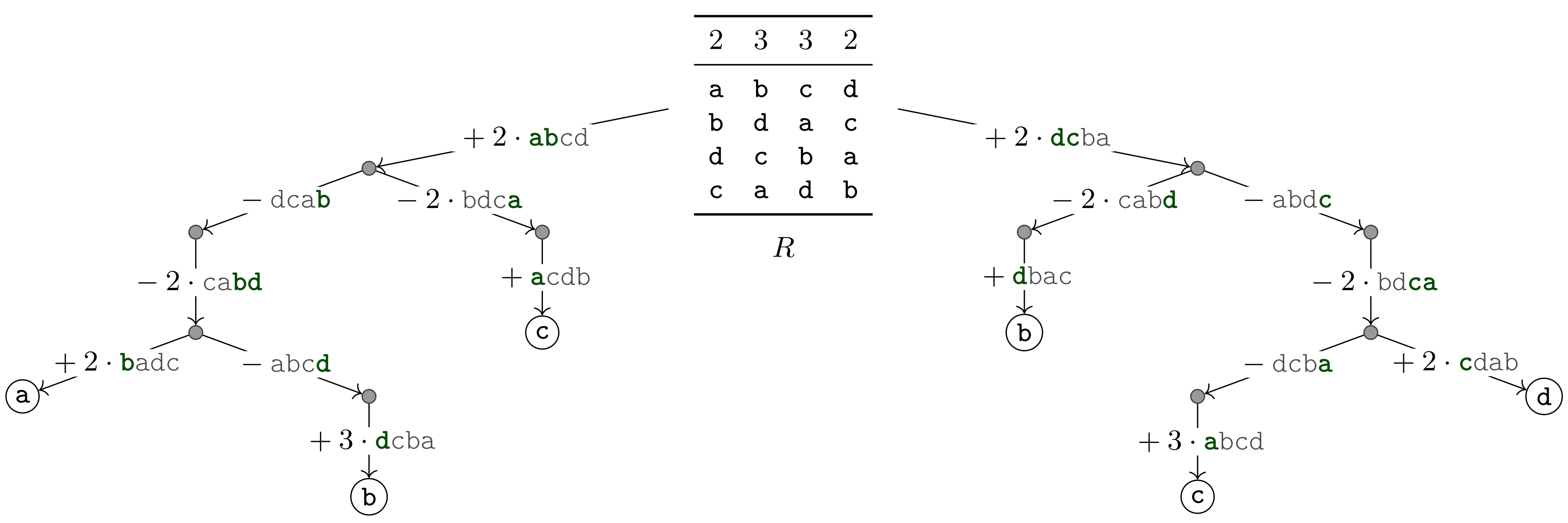
Set-valued voting rules with the optimistic and pessimistic set extensions (i.e., voters like a set according to the best/worst alternative in it). Our impossibility results are significant improvements over prior work: for the pessimistic extension, the previous result needs 971 voters!

We also show that no probabilistic voting rule can be Condorcet-consistent and satisfy *strong SD-participation*, answering an open question (Brandl et al., AAMAS 015).

For at least **12 voters** and at least **4 alternatives**, there does not exist a Condorcet-consistent voting rule that satisfies participation

For our encoding with exactly 4 alternatives (a, b, c, d) the SAT solver returned *unsatisfiable*. Together with a (manually-produced) inductive step, we deduce an impossibility theorem for arbitrary number of alternatives.

We then extract a *minimal unsatisfiable set* (MUS) which allows extracting a human-readable proof.



The proof shown here (and the other proofs in the paper) exhibit a curious symmetry:

The initial profile R is invariant under relabelling alternatives by abcd ↦ dcba. Thus, the left-hand half of the proof is symmetric to the right-hand half. This efficient style of proof was discovered by the computer; previous proofs discovered by humans are asymmetric.

All our impossibility proofs are presented as **proof diagrams** generated from an MUS.
A novel way to graphically represent impossibility proofs in social choice.

How to read the diagram:

$R - + \text{abcd} \longrightarrow R'$	profile R' is obtained from R by adding a voter with preferences abcd. If any of the **green bold** alternatives is selected at R, then one must be selected at R' by participation.
(a)	profile which admits Condorcet winner a.

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