Kemeny is NP-hard for 7 Voters

Dominik Peters
University of Oxford

Kemeny Rank Aggregation

Introduced by Kemeny (1959) and popularised by Young (1978-95).

Input: a profile ($\succ_i$) of strict preference rankings
Output: an optimal consensus ranking $\succ$ maximising agreement with the input profile, that is

$$\sum_{i \in N} |\succ_i \cap \succ| = \max_{\succ' \in \mathcal{L}(A)} \sum_{i \in N} |\succ_i \cap \succ'|.$$

Equivalently, we wish to find a ranking $\succ$ of minimal Kendall-tau distance to the profile.

Known Complexity Results

Easy for 1 or 2 voters (constant time)

NP-compl. for unbounded number of voters [Bartholdi III et al. 1989]

NP-compl. for 4 voters, 6 voters, 8 voters, ... [Dwork et al. 2001]

Finding a Kemeny *winner* is Θ_2^P-complete [Hemaspaandra et al. 2005]

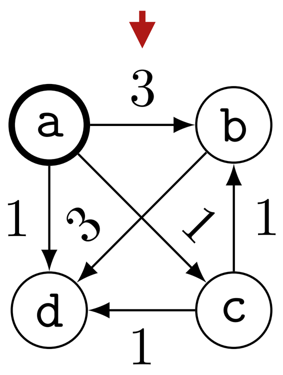
Open for 3 voters, 5 voters, 7 voters, 9 voters, ...

Weighted Majority Tournaments

Every profile can be associated with a weighted tournament on the alternative set, where arcs are labelled with the majority margin:

$$m_R(x,y) = \big|\{i \in N : x \succ_i y\}\big| - \big|\{i \in N : y \succ_i x\}\big|.$$

a	b	c	c	d
b	d	a	a	a
c	a	b	b	c
d	c	d	d	b



Kemeny is equivalent to finding a min-weight *feedback arc set* of this tournament: find as few arcs as possible that need to be turned around to make a transitive tournament. If there is an odd number of voters, then all arcs receive a non-0 majority margin — this makes the problem hard to reason about.

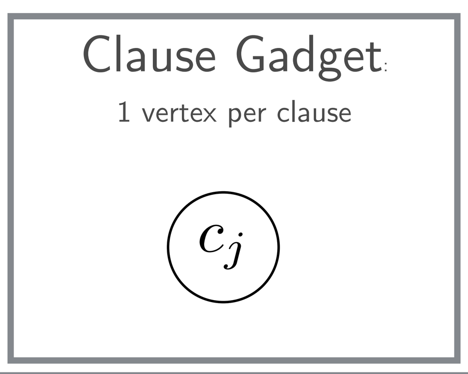
Theorem (Conitzer 2006): Feedback arc set is NP-complete on tournaments where every arc has weight 1.

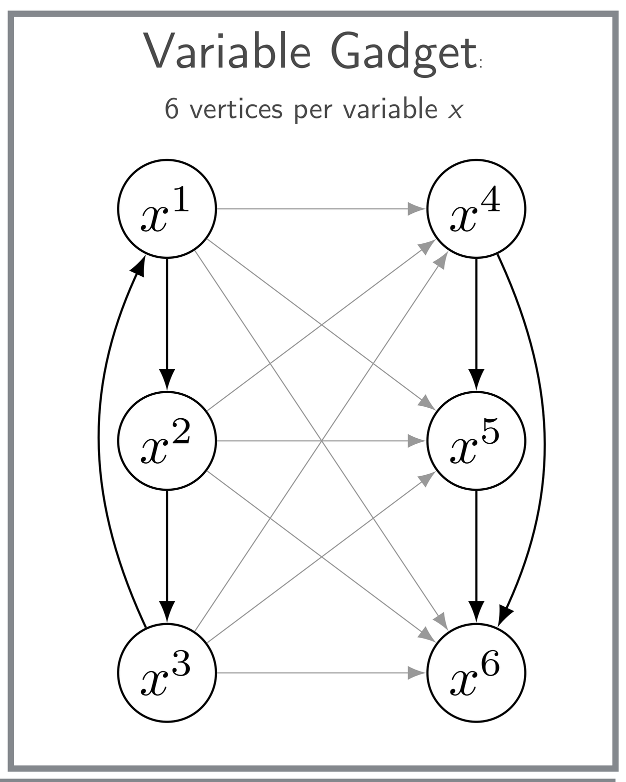
Our proof strategy: Show that all tournaments arising as hard instances of Conitzer's reduction can be induced by 7-voter profiles (a technique due to Bachmann et al. 2016)

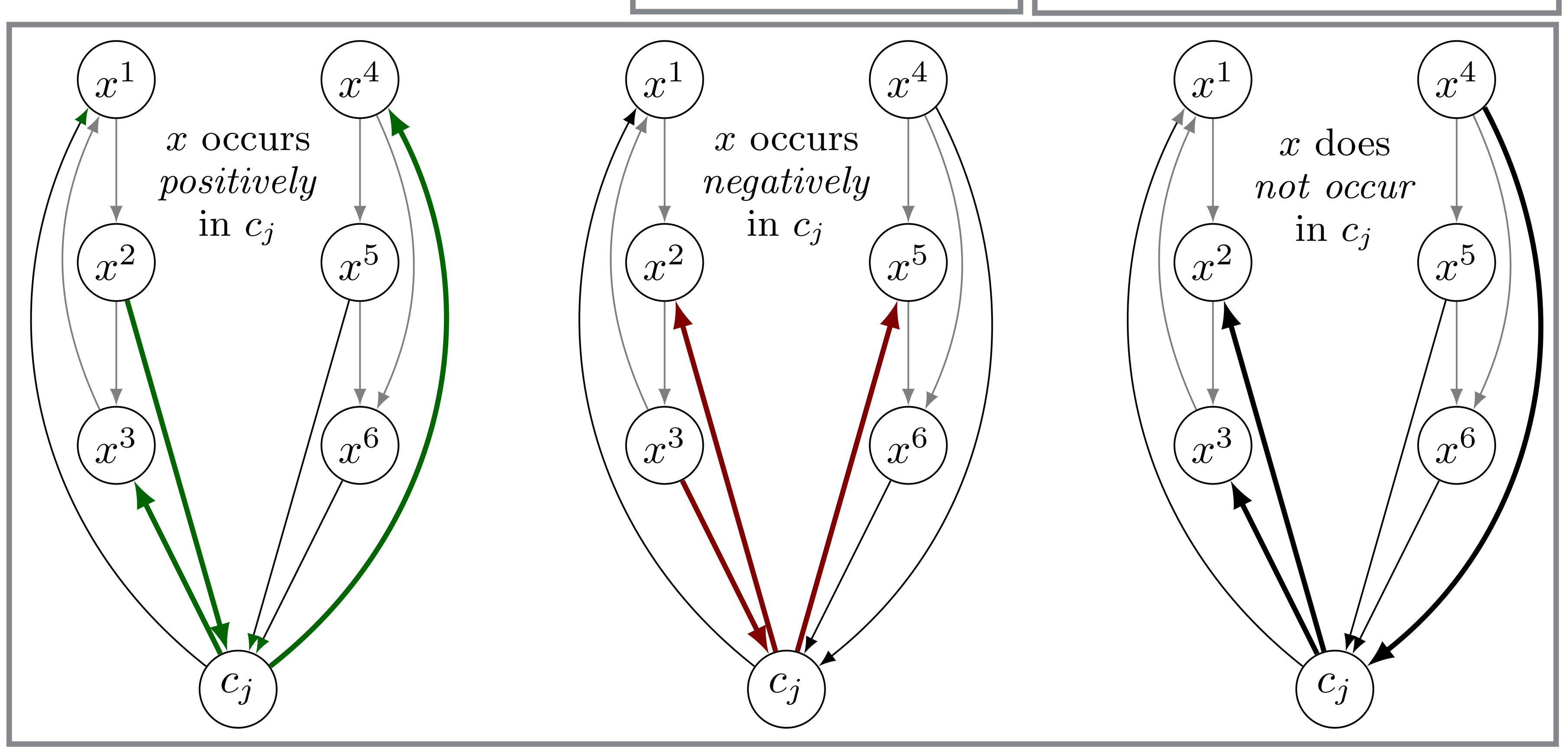
Conitzer's Reduction

Reduction from SAT. Given a formula φ, produce the following tournament with arc weights 1. Each variable gets a 6-vertex gadget, and a clause gadget is connected to all variable gadgets depending on whether that variable occurs positively, negatively, or does not occur.

[Technical Note: if x occurs negatively in a clause c_j, Conitzer's original reduction draws an arc from c_j to x^4. We draw the arc from c_j to x^5. Conitzer's correctness argument works even with this change.]







Proof.

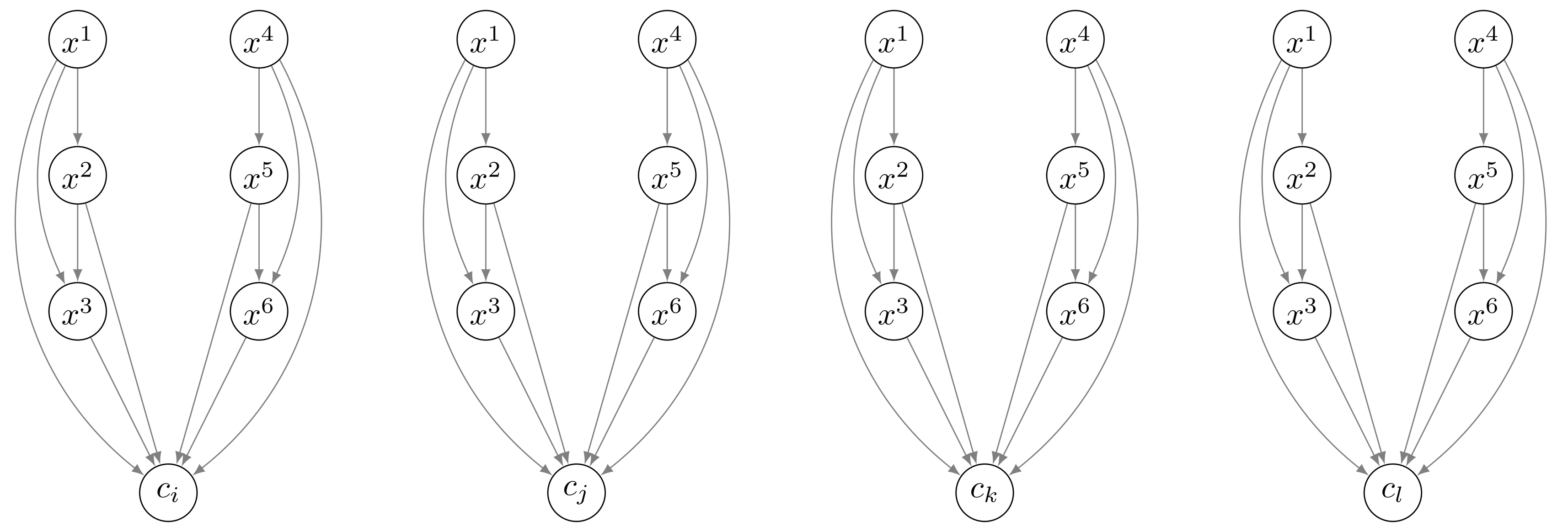
We may assume that the input formula φ satisfies the following:

- every clause contains exactly 2 or 3 distinct literals
- every variable occurs exactly once (positively or negatively) in a 3-clause
- every variable occurs exactly once positively in a 2-clause
- every variable occurs exactly once negatively in a 2-clause

SAT remains hard for such formulas.

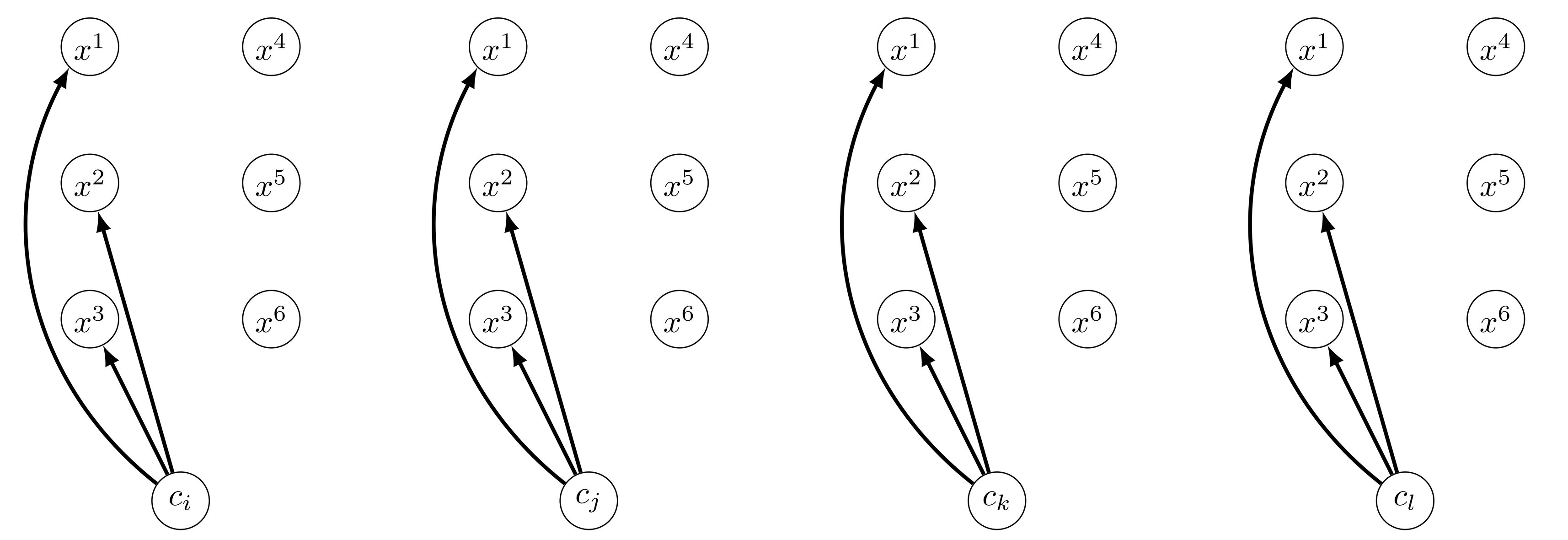
Let $x_1, \dots, x_n$ be the variables of ϕ, let C_2 be the set of 2-clauses, and let C_3 be the set of 3-clauses. The clauses are canonically ordered, $\overrightarrow{C}$. We denote the reverse ordering by $\overleftarrow{C}$. When specifying a voter, we let $\overrightarrow{\textsf{rest}}$ be some ordering of the vertices not yet mentioned in the vote.

Conitzer's tournaments can be induced by the following 7 voters:



voter 1: $(x_1^1\, x_1^2\, x_1^3\, x_1^4\, x_1^5\, x_1^6) \;\dots\; (x_n^1\, x_n^2\, x_n^3\, x_n^4\, x_n^5\, x_n^6) \;\; \overrightarrow{C_2} \;\; \overrightarrow{C_3}$

+



voter 2: $(x_1^1\, x_1^2\, x_1^3) \;\dots\; (x_n^1\, x_n^2\, x_n^3) \;\; \overrightarrow{C_2} \;\; \overrightarrow{C_3} \;\; \overrightarrow{\textsf{rest}}$

voter 3: $\overleftarrow{\textsf{rest}} \;\; (x_n^3\, x_n^2\, x_n^1) \;\dots\; (x_1^3\, x_1^2\, x_1^1) \;\; \overleftarrow{C_3} \;\; \overleftarrow{C_2}$

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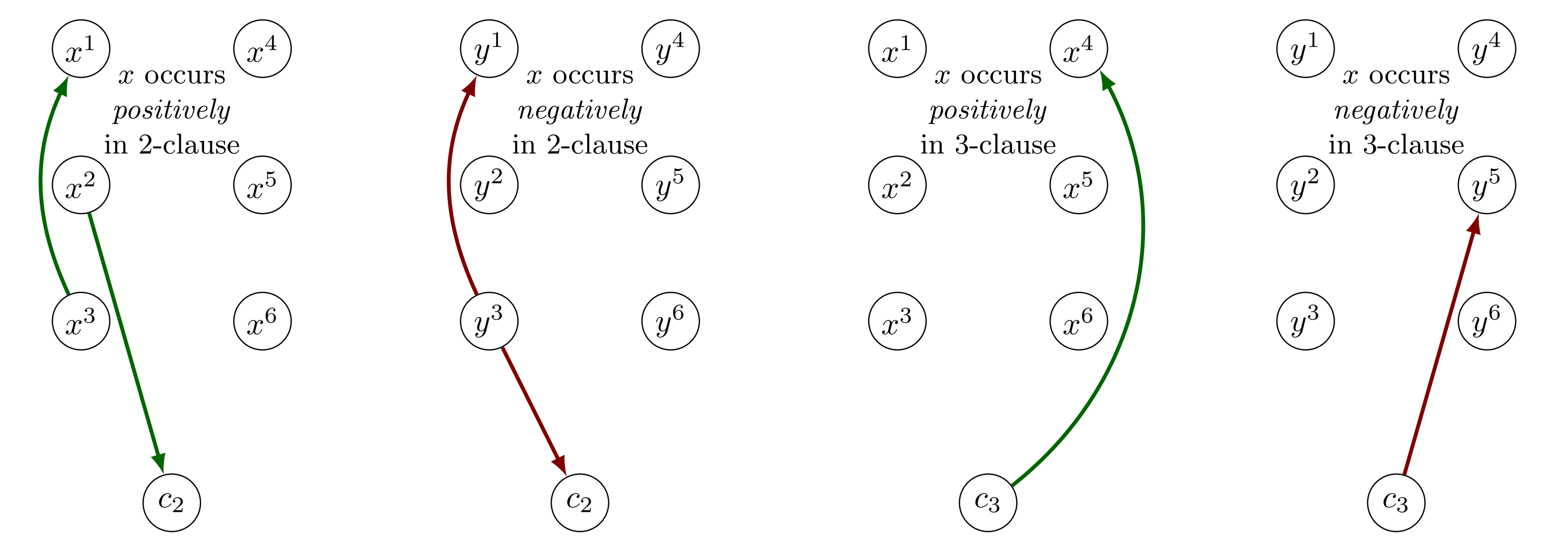

voter 4: $\Big(c_3\, x^4\, y^5\, z^4 \;\Big|\; c_3 = (x \vee \neg y \vee z) \in \overrightarrow{C_3}\Big) \;\; \Big(x^2\, y^3\, c_2\, y^1 \;\Big|\; c_2 = (x \vee \neg y) \in \overrightarrow{C_2}\Big) \;\; \overrightarrow{\textsf{rest}}$

voter 5: $\overleftarrow{\textsf{rest}} \;\; \Big(y^3\, y^1\, x^2\, c_2 \;\Big|\; c_2 = (x \vee \neg y) \in \overleftarrow{C_2}\Big) \;\; \Big(c_3\, z^4\, y^5\, x^4 \;\Big|\; c_3 = (x \vee \neg y \vee z) \in \overleftarrow{C_3}\Big)$

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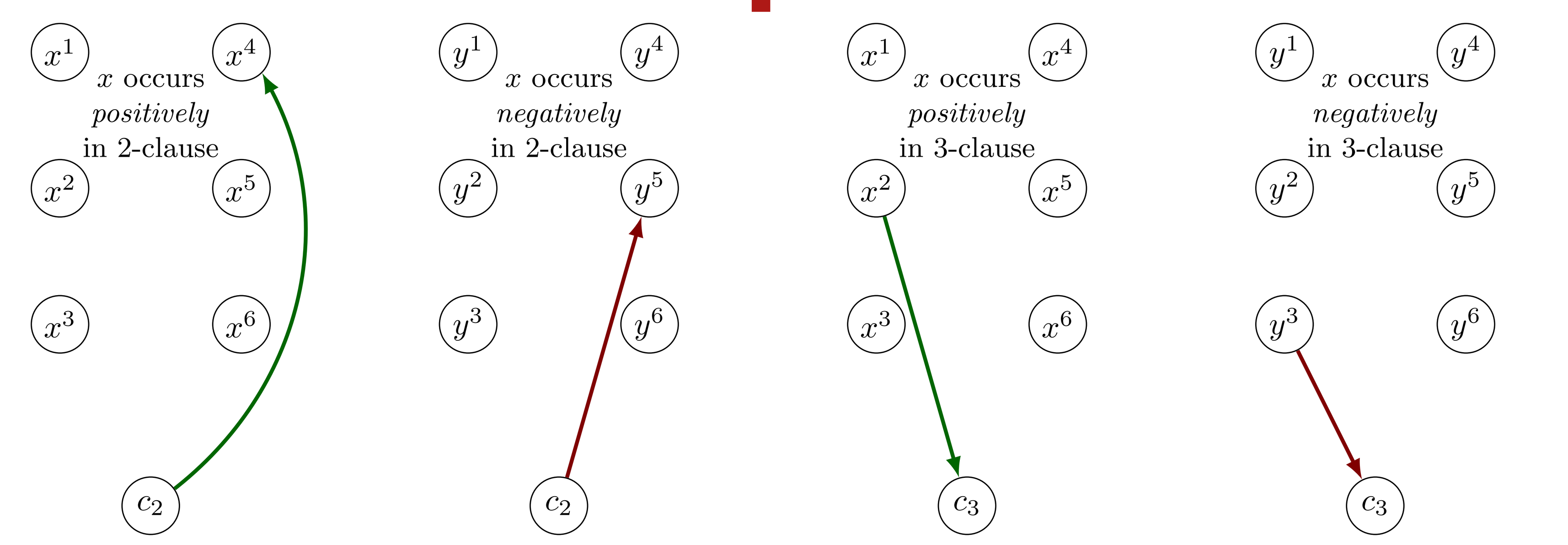

voter 6: $\Big(c_2\, x^4\, y^5 \;\Big|\; c_2 = (x \vee \neg y) \in \overrightarrow{C_2}\Big) \;\; \Big(x^2\, y^3\, z^3\, c_3 \;\Big|\; c_3 = (x \vee \neg y \vee \neg z) \in \overrightarrow{C_3}\Big) \;\; \overrightarrow{\textsf{rest}}$

voter 7: $\overleftarrow{\textsf{rest}} \;\; \Big(z^3\, y^3\, x^2\, c_3 \;\Big|\; c_3 = (x \vee \neg y \vee \neg z) \in \overleftarrow{C_3}\Big) \;\; \Big(c_2\, y^5\, x^4 \;\Big|\; c_2 = (x \vee \neg y) \in \overleftarrow{C_2}\Big)$

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