Optimized Distortion and Proportional Fairness in Voting

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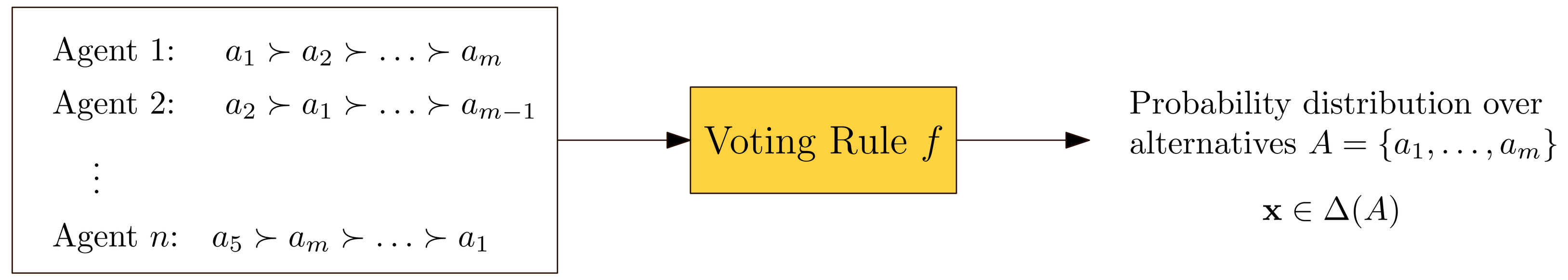
Single-Winner Randomized Voting

We consider the problem of designing randomized voting rules that aggregate agents' ranked preferences and arrive at a collective decision with *high social welfare* and which is *fair* to all agents.

- *Input:* Complete rankings over m alternatives A
- *Output:* Probability distribution over A



Implicit Utilitarian Voting: [Procaccia and Rosenschein, 2006]

Assume that each agent i has a cardinal utility function $u_i : A \to \mathbb{R}_{\geq 0}$ over alternatives, but **reports only the induced ranking** over alternatives to the voting rule.

Approach: We evaluate how well voting rules do, in **worst-case** scenario, on measures of social welfare (distortion) and of proportional fairness, computed based on the hidden utility functions.

Distortion

The *Utilitarian Welfare* of distribution $\mathbf{x} \in \Delta(A)$ is defined as the **sum of agent utilities**:

$$\mathrm{UW}(\mathbf{x}, \vec{u}) = \sum_{i \in N} u_i(\mathbf{x}), \quad \text{where} \quad u_i(\mathbf{x}) = \mathbb{E}_{a \sim \mathbf{x}}[u_i(a)].$$

Definition 1 (Distortion). The distortion of a voting rule f over the utility class $\mathcal{U}$ is

$$\mathrm{Dist}_m(f, \mathcal{U}) = \max_{\substack{\text{utility profiles } \vec{u} \in \mathcal{U}^n \\ \text{preference profile } \vec{\sigma} \text{ induced by } \vec{u}}} \frac{\max_{\mathbf{y} \in \Delta(A)} \mathrm{UW}(\mathbf{y}, \vec{u})}{\mathrm{UW}(f(\vec{\sigma}), \vec{u})},$$

i.e., the worst-case **ratio between the welfare of the optimal outcome and the welfare of** $f(\sigma)$.

We analyze distortion over different **utility classes:**

- **Unit-sum:** $\sum_{a \in A} u(a) = 1$.
- **Unit-range:** $\max_{a \in A} u(a) = 1$.
- **Approval:** $u_i(a) \in \{0, 1\}$ for all $a \in A$, with at least one approval. (**Subclass** of unit-range.)
- Balanced: $\max_{a \in A} u(a) \leq 1$ and $\sum_{a \in A} u(a) \geq 1$. (**Includes** unit-sum, unit-range, approval.)

Optimal Distortion

Theorem 1. For all classes of **unit-sum**, **unit-range**, **approval**, and **balanced** utilities, the *Stable Lottery rule* achieves the optimal $\Theta(\sqrt{m})$ distortion.

Prior to this work, [Boutilier et al., 2015] proposed a rule that achieves $O(\sqrt{m} \log^*(m))$ distortion for **unit-sum** utilities.

Stability in Multi-Winner Voting

In multi-winner voting, the problem of selecting a committee $X \subseteq A$ of k alternatives, a general notion of fairness is **stability**.

Idea: A group of $\frac{n}{k}$ agents should be able to decide over one of the k slots in the committee.

For a committee X and an alternative a, define $V(X, a) = |\{i : i \text{ prefers } a \text{ to all of } X\}|$.

Stable Committee: X is stable if for all $a \in A$, $V(a, X) < \frac{n}{k}$.

Stable Lottery: A distribution $\mathbf{X}$ over committees of size k is stable if for all alternatives $a \in A$,

$$\mathbb{E}_{X \sim \mathbf{X}}[V(a, X)] < \frac{n}{k}.$$

For every preference profile and any k, *a stable lottery always exists* [Cheng et al., 2020].

Stable Lottery Rule

Stable Lottery Rule. Let $\mathbf{X}$ be a **stable lottery** over committees of size $k = \sqrt{m}$. Then,

- With probability 1/2:
  1. Sample a committee $X \sim \mathbf{X}$
  2. Choose an alternative uniformly at random from X
- With probability 1/2:
  1. Choose an alternative uniformly at random from A

Proportional Fairness

Definition 2 (Proportional Fairness). The proportional fairness of a voting rule f is

$$\mathrm{PF}_m(f) = \max_{\vec{\sigma} \text{ induced by } \vec{u}} \max_{\mathbf{y} \in \Delta(A)} \frac{1}{n} \sum_{i \in N} \frac{u_i(\mathbf{y})}{u_i(f(\vec{\sigma}))},$$

i.e., the maximum possible **average multiplicative increase in agent utilities** when moving from $f(\vec{\sigma})$ to any other $\mathbf{y}$.

- First proposed in communication networks [Kelly et al., 1998].
- Scale-invariant: value stays the same if we rescale utility functions.

Optimal Proportional Fairness

Theorem 2. There exists a voting rule which is $O(\log m)$-proportionally fair.

- Our existence proof uses the minimax theorem for zero-sum games.

Theorem 3. This bound is optimal, as there are preference profiles for which any distribution has no approximation better than $\Omega(\log m)$.

Theorem 4. Given a preference profile, we can **compute in polynomial time** a distribution with **an (almost) optimal approximation** to proportional fairness.

Proportional Fairness ⇒ Nash Welfare Distortion

Nash Welfare Distortion. We also study the Nash welfare, which is the geometric mean of agent utilities: $\mathrm{NW}(\mathbf{x}, \vec{u}) = \left(\prod_{i \in N} u_i(\mathbf{x})\right)^{1/n}$.

We can define **distortion with Nash welfare** by replacing UW in Definition 1 with NW.

- It is well-known that $\mathrm{Dist}^{\mathrm{NW}}_m(f) \leq \mathrm{PF}_m(f)$.

  $\implies$ there exists a voting rule f with

  $$\mathbf{Dist}^{\mathbf{NW}}_{\boldsymbol{m}}(\boldsymbol{f}) \leq \boldsymbol{O}(\log \boldsymbol{m}).$$

- Our lower bound is at most $e \approx 2.718$.

Open Problem: Is constant distortion with Nash welfare achievable?

Truthful Voting Rules

Harmonic Rule [Boutilier et al., 2015]:

- With probability 1/2:
  1. Select an *agent* i uniformly at random
  2. Choose agent i's k^{th} most preferred alternative with probability $\propto 1/k$
- With probability 1/2:
  1. Choose an alternative uniformly at random from A

	Distortion, unit-sum	Distortion, unit-range	Proportional fairness
Harmonic Rule (truthful)	$\Theta(\sqrt{m \log m})$ ⋆	$\boldsymbol{\Theta(m^{2/3} \log^{1/3} m)}$	$\boldsymbol{\Theta(\sqrt{m \log m})}$
Best possible, truthful	$\Theta(\sqrt{m \log m})$ †	$\Theta(m^{2/3})$ ‡	$\boldsymbol{\Omega(\sqrt{m}), O(\sqrt{m \log m})}$

⋆ [Boutilier et al., 2015], † [Bhaskar et al., 2018], ‡ [Filos-Ratsikas et al., 2014, Lee 2013]

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