Party Approval PAV Converges to Nash

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Let N be a set of n voters and C a set of m candidates (or parties). An *approval profile* is a collection $A = (A_1, \dots, A_n)$ of non-empty subsets $A_i \subseteq C$ of approved parties for each voter $i \in N$.

Given a number of seats k, a *seat allocation* is a vector $x = (x_1, \dots, x_m)$ with $x_i \in \mathbb{N}$ for each party $i \in C$ such that $\sum_{i=1}^m x_i = k$. Given an approval profile A, and a number of seats k, the *PAV score* of a seat allocation x is defined as

$$\mathrm{PAV}(A, x) = \sum_{i \in N} H(\textstyle\sum_{j \in A_i} x_j),$$

where $H(t) = \sum_{j=1}^t \frac{1}{j}$ is the harmonic number. The *PAV rule* selects a seat allocation x that maximizes the PAV score. Note that we are operating in the "party-approval" setting where each party can receive an arbitrary number of seats.

Let $\Delta(C) = \{p \in [0,1]^m \mid \sum_{i=1}^m p_i = 1\}$ be the simplex of probability distributions over C. The *Nash rule* selects a distribution $p \in \Delta(C)$ that maximizes the log Nash product

$$\mathrm{Nash}(A, p) = \sum_{i \in N} \log(\textstyle\sum_{j \in A_i} p_j).$$

Note the following standard inequality connecting the harmonic and log functions:

$$\ln(t) + \gamma \leq H(t) \leq \ln(t) + \gamma + \frac{1}{2t}, \tag{1}$$

where γ is the Euler-Mascheroni constant.

We can now show that the PAV rule converges to the Nash rule as $k \to \infty$.

Theorem 1. *Let A be an approval profile. For each $k \in \mathbb{N}$, let x^k be the PAV allocation for A with k seats. Let $p^k = \frac{x^k}{k}$ be the distribution over C induced by x^k. Since $\Delta(C)$ is compact, the sequence p^k has a limit point $p^\infty \in \Delta(C)$. Let p^* be the Nash distribution for A. Then $\mathrm{Nash}(A, p^\infty) = \mathrm{Nash}(A, p^*)$.*

Proof. Suppose for a contradiction that $\mathrm{Nash}(A, p^\infty) < \mathrm{Nash}(A, p^*)$. Then there exists a rational distribution q and some $\varepsilon > 0$ such that $\mathrm{Nash}(A, q) > \mathrm{Nash}(A, p^\infty) + \varepsilon$.

Because the Nash objective is continuous and $p^k \to p^\infty$, we have $\mathrm{Nash}(A, p^k) \to \mathrm{Nash}(A, p^\infty)$. Thus, for sufficiently large k, we have

$$\mathrm{Nash}(A, p^k) < \mathrm{Nash}(A, q) - \varepsilon/2. \tag{2}$$

Now, choose some k^* large enough such that

- (2) holds,

- $k^* > n^2/\varepsilon$,
- k^* is a multiple of n, and
- k^* is a multiple of the denominators of the rational numbers $(q_j)_{j \in C}$.

From the last property, there exists a seat allocation y such that $q_j = y_j/k^*$ for all $j \in C$. We will show that y has a higher PAV score than $x := x^{k^*}$, a contradiction.

For each $i \in N$, write $u_i = (\sum_{j \in A_i} x_j)/k^*$ for their utility under x as a fraction of seats, and $u'_i = (\sum_{j \in A_i} y_j)/k^* = \sum_{j \in A_i} q_j$ for their utility under y and q.

Recall that PAV satisfies EJR. Let $i \in N$ be a voter. Note that $S = \{i\}$ forms a group of size $|S| \geq \ell \cdot \frac{n}{k^*}$ with $\ell = \lfloor \frac{k^*}{n} \rfloor$, and S is obviously cohesive as approving at least one common party. Thus, EJR guarantees that

$$k^* u_i \geq \left\lfloor \frac{k^*}{n} \right\rfloor = \frac{k^*}{n}, \tag{3}$$

where $k^* u_i$ is the number of seats in x going to approved candidates, and where we can remove the floor because k^* is a multiple of n.

Now we have

$$\begin{aligned}
\mathrm{PAV}(A, x) &= \sum_{i \in N} H(k^* u_i), \\
&\leq \sum_{i \in N} \ln(k^* u_i) + \gamma + \frac{1}{2k^* u_i} && \text{(by (1))} \\
&\leq \sum_{i \in N} \ln(k^* u_i) + \gamma + \frac{n}{2k^*} && \text{(by (3))} \\
&< \sum_{i \in N} \ln(k^* u_i) + \gamma + \frac{\varepsilon}{2n} && \text{(since } k^* > n^2/\varepsilon) \\
&= \mathrm{Nash}(A, p^k) + \sum_{i \in N} \ln(k) + \gamma + \frac{\varepsilon}{2n} \\
&< \mathrm{Nash}(A, q) - \frac{\varepsilon}{2} + \sum_{i \in N} \ln(k^*) + \gamma + \frac{\varepsilon}{2n} && \text{(from (2))} \\
&< \mathrm{Nash}(A, q) - \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + \sum_{i \in N} \ln(k^*) + \gamma \\
&= \sum_{i \in N} \ln(k^* u'_i) + \gamma \\
&\leq \sum_{i \in N} H(k^* u'_i) && \text{(by (1))} \\
&= \mathrm{PAV}(A, y),
\end{aligned}$$

which is the desired contradiction. □