COMSOC Lecture 4: Apportionment and Approval-Based Committee Elections Part 2

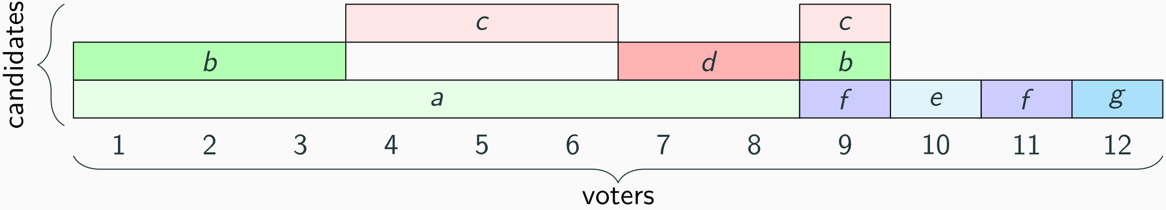
Dominik Peters

CNRS, LAMSADE, Université Paris Dauphine - PSL

IASD · 2025-01-29

Approval-based Committee Elections (ABC)

- A set C of candidates, k of which have to be elected.
- A set N of n voters.
- Each voter $i \in N$ approves a subset $A_i \subseteq C$; we get a profile $P = (A_i)_{i \in N}$.
- Outcome: committee $W \subseteq C$ with $|W| = k$.
- We say that i's utility is $u_i(W) = |W \cap A_i|$.



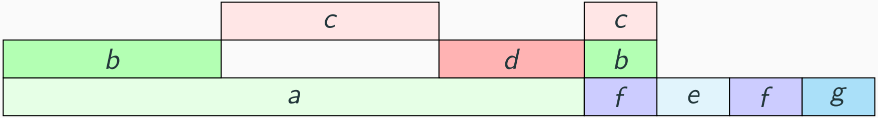
$A_1 : \{a, b\}$ $A_2 : \{a, b\}$ $A_3 : \{a, b\}$ $A_4 : \{a, c\}$

$A_5 : \{a, c\}$ $A_6 : \{a, c\}$ $A_7 : \{a, d\}$ $A_8 : \{a, d\}$

$A_9 : \{b, c, f\}$ $A_{10} : \{e\}$ $A_{11} : \{f\}$ $A_{12} : \{g\}$.

Multiwinner Approval Voting (AV)

Multi-Winner Approval Voting (AV) selects the k candidates which are approved by most voters. Formally, the AV-score of an alternative $c \in C$ is defined as $s(c) = |\{i \in N : c \in A_i\}|$ and AV selects committees W that maximise $\sum_{c \in W} s(c)$.



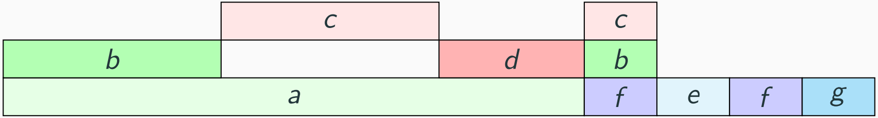
candidate	a	b	c	d	e	f	g
AV-score	8	4	4	2	1	2	1

With $k = 4$, AV selects either $\{a, b, c, d\}$ or $\{a, b, c, f\}$. Note that the former leaves 3 voters without representation; the latter leaves 2 voters without representation.

Chamberlin–Courant (CC)

This rule is almost the opposite of AV!

The CC rule outputs all committees W that maximise $|\{i \in N : A_i \cap W \neq \emptyset\}|$: it maximizes the number of represented voters.



With $k = 4$, CC selects $W = \{a, e, f, g\}$ since this is the unique committee where every voter approves at least one member.

While this committee indeed provides some satisfaction for every voter, it includes alternatives (e and g) that are approved only by single voters.

Thiele methods

CC is an example of a rule in the family of Thiele methods, introduced by Thorvald N. Thiele (1838–1910), a Danish astronomer and mathematician.

A Thiele method is a rule selecting committees that maximize total voter satisfaction, for some definition of "satisfaction" that is a function of utility.

Definition

An ABC voting ruje f is a *Thiele method* if there exists a non-decreasing function $w : \{0, 1, 2, \dots\} \to \mathbb{R}_{\geqslant 0}$ such that $f(P) = \arg\max_{W \subseteq C : |W| = k} \sum_{i \in N} w(u_i(W))$.

CC is a Thiele method with $w(0) = 0$ and $w(t) = 1$ for all $t \geqslant 1$.

Question

Is AV a Thiele method?

Thiele methods

CC is an example of a rule in the family of Thiele methods, introduced by Thorvald N. Thiele (1838–1910), a Danish astronomer and mathematician.

A Thiele method is a rule selecting committees that maximize total voter satisfaction, for some definition of "satisfaction" that is a function of utility.

Definition

An ABC voting ruje f is a *Thiele method* if there exists a non-decreasing function $w : \{0, 1, 2, \dots\} \to \mathbb{R}_{\geqslant 0}$ such that $f(P) = \arg\max_{W \subseteq C : |W| = k} \sum_{i \in N} w(u_i(W))$.

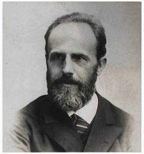
CC is a Thiele method with $w(0) = 0$ and $w(t) = 1$ for all $t \geqslant 1$.

Question

Is AV a Thiele method?

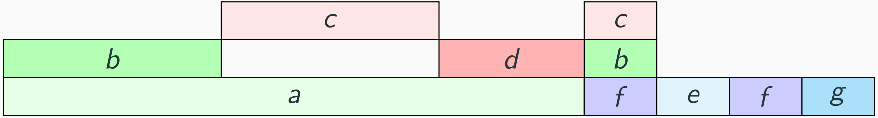
Yes: $\sum_{c \in W} |\{i \in N : c \in A_i\}| = \sum_{c \in W} \sum_{i \in N : c \in A_i} 1 = \sum_{i \in N} \sum_{c \in A_i \cap W} 1 = \sum_{i \in N} u_i(W)$. So we can take $w(t) = t$.

Proportional Approval Voting (PAV)



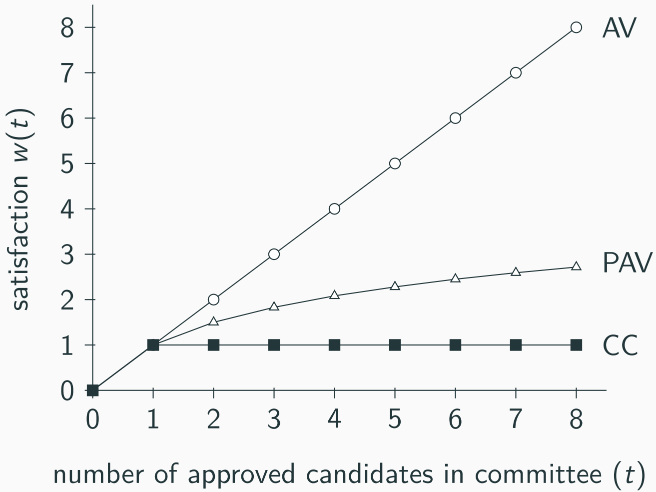
Thiele [1895] introduced what is today called Proportional Approval Voting (PAV). It is the Thiele method with

$$w(t) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{t}.$$



PAV selects the committee $W = \{a, b, c, f\}$.

Comparison of Thiele methods

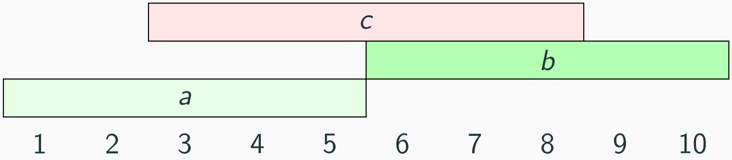


Some properties

- *Pareto optimality*: If $f(P) = W$, then there should be no other committee W' such that $u_i(W') \geqslant u_i(W)$ for all $i \in N$, with strict inequality for at least one agent.
  - AV and PAV are Pareto optimal (why?).
  - CC always selects at least one Pareto-optimal committee, possibly among others. To make it Pareto-optimal, can use the Thiele method with $w = (0, 1, 1 + \varepsilon, 1 + 2\varepsilon, \dots)$.
- *Committee monotonicity*: the analog of *house monotonicity*. The committee selected for size $k + 1$ should be a superset of the committee selected for size k. Q: Which rules satisfy it?

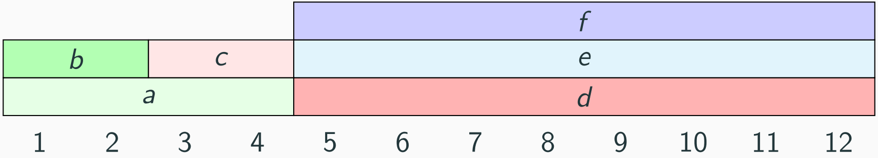
Some properties

- *Pareto optimality*: If $f(P) = W$, then there should be no other committee W' such that $u_i(W') \geqslant u_i(W)$ for all $i \in N$, with strict inequality for at least one agent.
  - AV and PAV are Pareto optimal (why?).
  - CC always selects at least one Pareto-optimal committee, possibly among others. To make it Pareto-optimal, can use the Thiele method with $w = (0, 1, 1 + \varepsilon, 1 + 2\varepsilon, \dots)$.
- *Committee monotonicity*: the analog of *house monotonicity*. The committee selected for size $k + 1$ should be a superset of the committee selected for size k. Q: Which rules satisfy it? AV satisfies it (it selects candidates "sequentially" by their approval score).



For $k = 1$, PAV and CC select $\{c\}$. For $k = 2$, they select $\{a, b\}$

Justified Representation



With $k = 3$, AV selects $\{d, e, f\}$.

Intuitively, this is unfair to the first 4 voters. They make up a third of the voters, so "deserve" a seat, and they agree on a common candidate a, but they are unrepresented. The axiom of Justified Representation (JR) says that this is not allowed to happen.

Definition

A committee satisfies *Justified Representation* (JR) if there does not exist a group $S \subseteq N$ of voters with $|S| \geqslant \frac{n}{k}$ and $\bigcap_{i \in S} A_i \neq \emptyset$ but $u_i(W) = 0$ for all $i \in S$.

Note on the definition

Definition

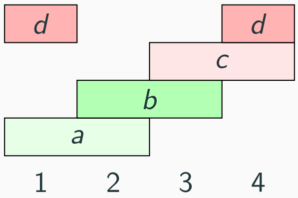
A committee satisfies *Justified Representation* (JR) if there does not exist a group $S \subseteq N$ of voters with $|S| \geqslant \frac{n}{k}$ and $\bigcap_{i \in S} A_i \neq \emptyset$ but $u_i(W) = 0$ for all $i \in S$.

An equivalent definition:

Definition

A committee satisfies *Justified Representation* (JR) if for every group $S \subseteq N$ of voters with $|S| \geqslant \frac{n}{k}$ and $\bigcap_{i \in S} A_i \neq \emptyset$ there exists at least one voter $i \in S$ with $u_i(W) > 0$.

This seems oddly weak, why don't we want that $W \cap \bigcap_{i \in S} A_i \neq \emptyset$?



CC satisfies JR

Recall that CC selects the committee maximizing the number of voters who approve at least one committee member. Let W be a committee selected by CC. Call a voter i covered if $A_i \cap W \neq \emptyset$, and let $s_{\mathrm{CC}}(W)$ be the number of covered voters. Thus, CC maximizes s_{CC}.

Theorem

Every CC committee satisfies JR.

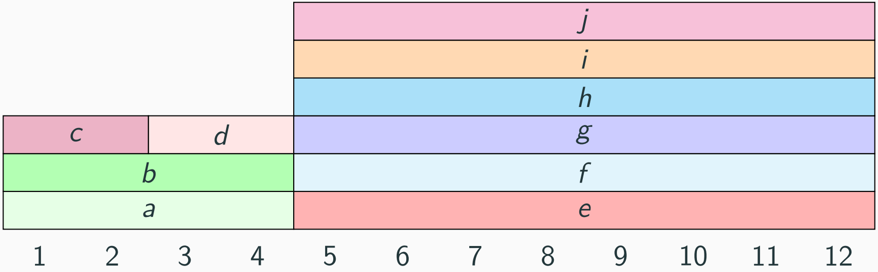
For a contradiction, suppose S with $|S| \geqslant \frac{n}{k}$ and $c^* \in \bigcap_{i \in S} A_i$, but none of $i \in S$ is covered.

Let $n' < n$ be the number of covered voters. For each candidate $c \in W$, say that contribution δ_c is the number of voters with $W \cap A_i = \{c\}$. Note that $\sum_{c \in W} \delta_c \leqslant n'$. Hence the average contribution of a committee member is $< \frac{n}{k}$. Thus, there exists a candidate $c^\dagger \in W$ with $\delta_{c^\dagger} < \frac{n}{k}$. But then

$$s_{\mathrm{CC}}(W \setminus \{c^\dagger\}) > s_{\mathrm{CC}}(W) - \frac{n}{k} \quad \text{and} \quad s_{\mathrm{CC}}(W \setminus \{c^\dagger\} \cup \{c^*\}) > s_{\mathrm{CC}}(W) - \frac{n}{k} + \frac{n}{k},$$

contradicting optimality of W. □

Extended Justified Representation



With $k = 6$, AV selects $\{e, f, g, h, i, j\}$. CC selects many committees, including $\{a, e, f, g, h, i\}$.

Again, intuitively, this is unfair to the first 4 voters. They make up a third of the voters, so "deserve" a third of the seats, i.e. 2 seats. They agree on two common candidates a and b, but none of them approve 2 candidates in these committees. The axiom of Extended Justified Representation (EJR) says that this is not allowed to happen.

Definition

A committee satisfies *Justified Representation* (JR) if there does not exist a group $S \subseteq N$ of voters with $|S| \geqslant \ell \cdot \frac{n}{k}$ and $|\bigcap_{i \in S} A_i| \geqslant \ell$ but $u_i(W) < \ell$ for all $i \in S$.

PAV satisfies EJR

Theorem

PAV satisfies EJR (and therefore also JR).

Let W be a PAV committee.

Assume for a contradiction that $S \subseteq N$ has size $|S| \geqslant \ell \cdot \frac{n}{k}$ and $u_i(W) < \ell$ for all $i \in S$, but there exists $c^* \in \bigcap_{i \in S} A_i \setminus W$.

Let $\tilde{W} = W \cup \{c^*\}$. (Note $|\tilde{W}| = k + 1$.) How much does the PAV score go up by adding c^*?

$$\text{PAV-score}(\tilde{W}) - \text{PAV-score}(W) \geqslant |S| \cdot \frac{1}{\ell} \geqslant \frac{n}{k}.$$

PAV satisfies EJR

Let $c \in \tilde{W}$. How much PAV-score do we lose if we remove c from $\tilde{W}$?

$$\text{PAV-score}(\tilde{W}) - \text{PAV-score}(\tilde{W} \setminus \{c\}) = \sum_{i \in N : c \in A_i} \frac{1}{u_i(\tilde{W})}.$$

What is the average loss?

$$\begin{aligned}\frac{1}{k+1} \sum_{c \in \tilde{W}} \Big(\text{PAV-score}(\tilde{W}) - \text{PAV-score}(\tilde{W} \setminus \{c\})\Big) &= \frac{1}{k+1} \sum_{c \in \tilde{W}} \sum_{i \in N : c \in A_i} \frac{1}{u_i(\tilde{W})} \\ &= \frac{1}{k+1} \sum_{i \in N} \sum_{c \in A_i \cap \tilde{W}} \frac{1}{u_i(\tilde{W})} \\ &= \frac{1}{k+1} \sum_{i \in N} 1 < \frac{n}{k}.\end{aligned}$$

Hence there exists some $c^\dagger \in \tilde{W}$ such that

$$\text{PAV-score}(\tilde{W}) - \text{PAV-score}(\tilde{W} \setminus \{c^\dagger\}) < \frac{n}{k}.$$

PAV satisfies EJR

So we have proved that

$$\text{PAV-score}(\tilde{W}) - \text{PAV-score}(W) \geqslant \frac{n}{k}.$$

and that there is $c^\dagger \in \tilde{W}$ such that

$$\text{PAV-score}(\tilde{W}) - \text{PAV-score}(\tilde{W} \setminus \{c^\dagger\}) < \frac{n}{k}.$$

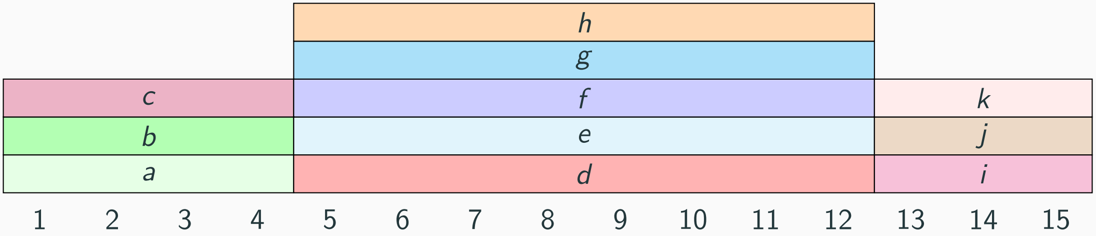
Hence

$$\begin{aligned}\text{PAV-score}(W) &\leqslant \text{PAV-score}(\tilde{W}) - \frac{n}{k} \\ &< \text{PAV-score}(\tilde{W} \setminus \{c^\dagger\}) + \frac{n}{k} - \frac{n}{k} \\ &= \text{PAV-score}(\tilde{W} \setminus \{c^\dagger\}),\end{aligned}$$

contradicting optimality of W. $\square$

PAV satisfies EJR

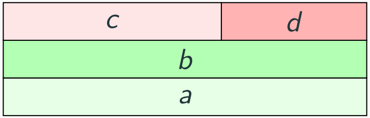
A party-list profile is one where every two voters either approve the same candidates or disjoint candidates: $A_i = A_j$ or $A_i \cap A_j = \emptyset$. For example:



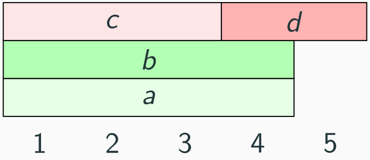
On these profiles, PAV implements D'Hondt, so for $k = 8$, it gives 2/5/1 seats to the "parties".

PAV is the only Thiele method that satisfies EJR. This follows from D'Hondt being the only divisor rule satisfying lower quota.

PAV is not strategyproof



For $k = 3$, PAV selects $\{a, b, c\}$.



For $k = 3$, PAV selects $\{a, b, d\}$. The 5th voter gains by underreporting!

Theorem

No voting rule satisfies JR, weak efficiency, and is strategyproof.

Aside: proving impossibility theorems using SAT solvers

Theorem

No voting rule satisfies JR, weak efficiency, and is strategyproof, even for $n = k = 3$ and $m = 4$.

How can one prove an impossibility theorem like this one?

One possibility is to use a computer, and in particular SAT solvers. These have many applications, like solving Sudoku puzzles, and finding voting rules that satisfy desired axioms is not very different from that.

Idea: enumerate all possible preference profiles P and all committees W, and make a new Boolean variable $\varphi_{P,W}$ saying that "rule f selects W at P". Then add clauses (constraints) like $\bigvee_W \varphi_{P,W}$ for each P (saying f must select at least one outcome) and $\neg\varphi_{P,W}$ whenever W fails JR.

Then ask a SAT solver if the resulting formula is satisfiable. If not $\rightarrow$ impossibility theorem!

PAV is NP-complete

Consider the following decision problem:

Instance:	Profile P, committee size k, target value B
Question:	Does there exist a committee W with $\|W\| = k$ and PAV-score$(W) \geqslant B$?

This problem is NP-complete.

Clearly it is in NP. For showing NP-hardness, we can reduce from Cubic Independent Set:

Instance:	Graph $G = (V, E)$ with $d(v) = 3$ for all $v \in V$, target size k
Question:	Does there exist $V' \subseteq V$ with $\|V'\| = k$ such that for each $e = \{u, v\} \in E$, either $u \notin V'$ or $v \notin V'$?

PAV is NP-complete

Let $G = (V, E)$ be a cubic graph and let $1 \leqslant k \leqslant |V|$.

Introduce candidates $C = V$, and voters $N = E$. Each voter approves its endpoints. Set $B = 3k$.

We prove: There is a committee W with PAV-score$(W) \geqslant B$ if and only if G has an independent set of size k.

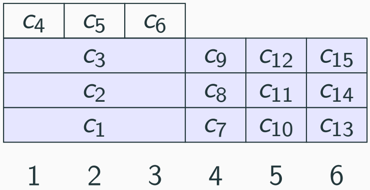
$(\Leftarrow)$: Let V' be an independent set of size k. Then no voter approves 2 candidates in V'. Each candidate in V' is approved by the 3 incident edges. So the PAV-score of V' is $3k$.

$(\Rightarrow)$: Suppose W has PAV-score $3k$. Each candidate is approved by 3 voters, so can contribute at most 3 to the PAV score. Since the total score is $3k$, each member of W contributes 3. This can only happen if no voter approves more than 1 candidate in W, so W is an independent set. $\square$

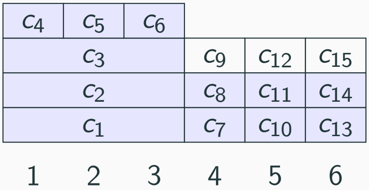
PAV can be computed in practice

In practice, PAV can be computed even for pretty large instances using Integer Linear Programming (ILP) solvers such as Gurobi.

Is PAV always right?



(a) PAV



(b) another outcome