

On Two Choice Rules Similar to the Plurality Rule and Borda's Rule

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§1. Introduction

Axiomatic definitions are given for two rules for group choice of alternatives. These rules coincide with the plurality rule (PL) and Borda's rule (BR) in the case of strict individual orderings of preference on a set X . The definition of PL for strict orderings of X is given in [7], and that of BR in [10] (see also [9]). In §2 we give axioms and definitions needed for formulating Theorems 1 and 2 in §3, some simple corollaries, and the definitions of PL and BR. In §3 we prove Theorems 1 and 2, which axiomatically characterize two rules, the first a generalization of PL, and the second a rule similar to BR. In §4 the rules defined in §3 are compared with PL and BR, and it is shown that they coincide in a special case. In §5 we comment on the axioms, the PL and BR rules, and the rules defined by Theorems 1 and 2.

§2. Axioms and definitions

Let $\{a, b, \dots, x, \dots\} = X$ be a finite and nonempty set of alternatives, with $|X| = m$. Let $X_{\succeq_j} = X_j$ be a well-ordering of the set X by a pre-order relation $\succeq_j$, i.e., $\succeq_j$ is a reflexive and transitive binary relation. As usual, $x \sim_j y$ means that $x \succeq_j y$ and $x \preceq_j y$, and $x \succ_j y$ means that $x \succeq_j y$ and $x \not\sim_j y$. We say that X_j is an *indifference* if $x \sim_j y$ for all $x, y \in X$, and X_j is a *strict preference* if $x \not\sim_j y$ or $x \neq y$.

We shall consider finite sets Π that are collections of orderings of X , i.e., $\Pi = \{X_1, \dots, X_n\}$. Let f be a mapping (choice rule) defined on the set of all finite Π and satisfying the condition

A0. $\emptyset \neq f(\Pi) \subset X$.

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Let h be a permutation on X and g a permutation on $\{1, \dots, n\}$ and let hX_j and g_i be the images of the mappings h and g , respectively, for orderings X_j and numbers $i \in \{1, \dots, n\}$. The symmetry axiom is as follows:

- A1. a) $f(h\Pi) = f(\{hX_1, \dots, hX_n\}) = hf(\Pi)$ for all h and Π ;
 b) $f(\Pi g) = f(\{X_{g_1}, \dots, X_{g_n}\}) = f(\Pi)$ for all g and Π .

The axiom A2, called separability in [8] and consistency in [7], [9], and [10], is as follows:

- A2. $f(\Pi) \cap f(\Pi^1) \neq \emptyset \Rightarrow f(\Pi \cup \Pi^1) = f(\Pi) \cap f(\Pi^1)$ for all Π and Π^1 .

Here $\Pi \cup \Pi^1 = \{X_1, \dots, X_n, X_1^1, \dots, X_p^1\}$, i.e., is the set formed from both the $X_j \in \Pi$ and the $X_j^1 \in \Pi^1$.

Denote by X_{j-} the ordering that is the reverse of X_j , i.e., $x \succ_j y \Leftrightarrow y \succ_{j-} x$, and let $\Pi^- = \{X_{i-}, \dots, X_{n-}\}$. The following axiom, apparently first discussed in [3], is called the duality axiom.

- A3. $f(\Pi) \neq X \Rightarrow f(\Pi) \subset X \setminus f(\Pi^-)$ for all Π .

Note that axiom A3 can be replaced by the following conditions if A0–A2 hold:

- A3*. $f(\Pi) \cap f(\Pi^-) \neq \emptyset \Rightarrow f(\Pi) = f(\Pi^-) = X$ for all Π .

- A3**. $f(\Pi \cup \Pi^-) = X$ for all Π .

LEMMA 1. *If f satisfies A0–A2, then A3, A3*, and A3** are equivalent.*

PROOF. $A3 \Rightarrow A3^*$ is obtained from A0, since

$$f(\Pi) \cap f(\Pi^-) \neq \emptyset \Rightarrow f(\Pi) \not\subset X \setminus f(\Pi^-) \Rightarrow f(\Pi) = f(\Pi^-) = X$$

$A3^* \Rightarrow A3^{**}$ follows from the fact that $\omega = \omega^-$ for $\omega = \Pi \cup \Pi^-$, and, consequently,

$$f(\omega) = f(\omega^-) = f(\Pi \cup \Pi^-) = X.$$

$A3^{**} \Rightarrow A3$. Suppose that $f(\Pi) \neq X$. Then in the case $f(\Pi) \cap f(\Pi^-) \neq \emptyset$ we have that

$$X = f(\Pi \cup \Pi^-) = f(\Pi) \cap f(\Pi^-) \neq X$$

in view of A2 and A3**. Consequently, $f(\Pi) \cap f(\Pi^-) = \emptyset$.

The next two axioms partially concern Pareto optimality and partially concern the concept of independence of alternatives. We use the notation: $a \succ_{\Pi} b \Leftrightarrow a \succeq_j b \forall X_j \in \Pi$ and $\exists X_r \in \Pi$ such that $a \succ_r b$.

Suppose that $X' \subset X$, and let X'_j be the restrictions of the orderings $X_j \in \Pi$. It is convenient to denote the collection $\Pi' = \{X'_1, \dots, X'_n\}$ by (Π, X') .

- A4. If there exists an x such that $x \succeq_{\Pi} y$, then $f(\Pi) = f(\Pi, X \setminus \{y\})$.

- A5. If $x \succeq_{\Pi} y$ for all x , then $f(\Pi) = f(\Pi, X \setminus \{y\})$.

Each ordering X_j can be represented by the quotient set $X/\sim_j = \{l_{j1}, \dots, l_{jm_j}\}$, $1 \leq m_j \leq m$ where $x \in l_{ji}$, $y \in l_{jk}$, $i > k \Leftrightarrow x \succ_j y$. We associate with each l_{jt} the number $\alpha_{jt} = \alpha(l_{jt})$, where the numbers α_{jt} satisfy:

$$\left. \begin{aligned} \alpha_{jt} &= \alpha_{rt} + \beta_j, \quad t \in \{i, \dots, m_j\}, \text{ if there exists a permutation } h \\ &\text{such that } X_j = hX_r \text{ and } \beta_j \text{ is any number.} \end{aligned} \right\} \quad (1)$$

We also need the notation

$$\alpha_{jt}(x) = \begin{cases} \alpha_{jt}, & x \in l_{jt}, \\ 0, & x \notin l_{jt}. \end{cases}$$

It follows from a lemma in [5] that for any positive integer N and any f satisfying A0–A2 there exist numbers α_{jt} satisfying not only (1) but also such that

$$f(\Pi) = \{x: \sum_{j=1}^n \sum_{t=1}^{m_j} \alpha_{jt}(x) = \max_{y \in X} \sum_{j=1}^n \sum_{t=1}^{m_j} \alpha_{jt}(y)\} \quad (2)$$

for all $\Pi = \{X_1, \dots, X_n\}$ such that $n \leq N$.

Note that in the case of strict orderings we get the plurality rule by setting $\alpha_{j1} = 1$ and $\alpha_{j2} = \dots = \alpha_{jm_j} = 0$ for $X_j \in \Pi$ in (2), and we get Borda's rule by setting $\alpha_{ji} = m - i$ in (2).

LEMMA 2. *Let X_1 be a strict ordering. If f satisfies axioms A0–A2 and A5, then (2)*

$$\alpha_{11} - \alpha_{1i} > 0, \quad 2 \leq i \leq m_1 = m,$$

for any N .

PROOF. Suppose that $\Pi = \{X_1, \dots, X_n\}$, $n \leq N$, $X_2, \dots, X_n$ are indifferencees, and $l_{11} = \{a\}$. By A0 and A5, $f(\Pi) = \{a\}$, and this implies the inequality to be proved, since $f(\Pi)$ is defined by (2). It is also obvious that $\alpha_{11} - \alpha_{1i} > 0$ does not depend on N .

§3. Two axiomatic definitions

THEOREM 1. *There exists a unique choice rule satisfying Axioms A0–A2 and A4; here $f(\Pi)$ is defined by setting $\alpha_{j1} = 1$ and $\alpha_{ji} = 0$, $2 \leq i \leq m_j$ in (2) for all $X_j \in \Pi$.*

PROOF. Suppose that for some $N \geq 2$ the numbers α_{jt} in (2) are known. Let Π be such that $X_1 \in \Pi$ is a strict ordering, $X_2 \in \Pi$ is an arbitrary ordering, and the $X_k \in \Pi$ are indifferencees for $2 \leq k \leq n \leq N$. By (1) and (2), we can let

$$\alpha_{j1} = 1 \quad \text{for all } X_j \in \Pi. \quad (3)$$

Moreover, suppose that $l_{11} = \{a\}$, $l_{12} = \{b\}$, $b \in l_{21}$ and $a \in l_{2i}$, where $i \in \{2, \dots, m_2\}$. Since $b \succeq_{\Pi} x$ for all $x \notin \{a, b\}$, it follows from A4 and A1 that $f(\Pi) = \{a, b\}$. From this, and from (1) and (2), we get that $\alpha_{11} + \alpha_{2i} = \alpha_{12} + \alpha_{21}$ which together with (3) gives us that

$$\alpha_{2i} = \alpha_{12}, \quad i \in \{2, \dots, m_2\}. \quad (4)$$

Note that the last equality does not depend on N . Therefore, by Lemma 2 and relations (3) and (4), it follows from the arbitrariness of X_2 and i that for any Π

$$i = \alpha_{j1} > \alpha_{ji} = \alpha, \quad i = 2, \dots, m_j \quad \text{for all } X_j \in \Pi.$$

This and (2) imply that f is unique. The second part of the assertion follows from the fact that α can be arbitrary if only $\alpha < 1$.

THEOREM 2. *There exists a unique choice rule satisfying axioms A0–A3 and A5; further, $f(\Pi)$ is determined by setting $\alpha_{ji} = m_j - i$, $1 \leq i \leq m_j$, in (2) for all $X_j \in \Pi$.*

PROOF. Suppose that $\Pi = \{X_1, X_2\}$ and

$$X_1 = a \succ b \succ x_1 \succ \cdots \succ x_m,$$

$$X_2 = l_{21} \succ \cdots \succ l_{2i-1} \succ l_{2i} \succ \cdots \succ l_{2m_2},$$

where $b \in l_{2i-1}$ and $a \in l_{2i}$.

Let the numbers α_{jt} determining $f(\Pi)$ according to (2) f or $N \geq 2$ correspond to the equivalence classes l_{jt} . In view of (1) the numbers α_{jt} do not change if we set $\{x_{v+i}, \dots, x_m\} = \bigcup_{j=i}^{m_2} l_{2j} \setminus \{a\}$. Denote by X'_1 and X'_2 the orderings

$$X'_1 = a \succ b \succ x_1 \succ \cdots \succ x_v, \quad X'_2 = l_{21} \succ \cdots \succ l_{2i-1} \succ a.$$

In view of A5 the numbers α'_{ij} determining $f(\{X'_s, X'_t\})$ according to (2) can be assumed equal to α_{ij} . Let

$$X'_3 = X'_{2-} = a \succ l_{2i-1} \succ \cdots \succ l_{21}.$$

By A3 and Lemma 1, we have that $f(\{X'_2, X'_3\}) = X' = \{a, b, x_1, \dots, x_v\}$, i.e., by (2),

$$\alpha_{2i-1} - \alpha_{2i} = \alpha_{3i} - \alpha_{32}. \quad (5)$$

Let X'_4 be obtained from X'_3 by transposing the alternatives a and b , i.e.,

$$X'_4 = b \succ l'_{2i-1} \succ \cdots \succ l_{21},$$

where $l'_{2i-1} = (l_{2i-1} \setminus \{b\}) \cup \{a\}$. Since $f(\{X'_3, X'_4\})$ contains the alternatives a and b by virtue of A5, it follows from (2) that

$$\alpha_{31} - \alpha_{32} = \alpha_{41} - \alpha_{42}. \quad (6)$$

But again by virtue of A5, $\{a, b\} \subset f(\{X'_1, X'_4\})$. Consequently,

$$\alpha_{11} - \alpha_{12} = \alpha_{41} - \alpha_{42}. \quad (7)$$

Comparing (5)–(7), we get that

$$\alpha_{11} - \alpha_{12} = \alpha_{2i-1} - \alpha_{2i}. \quad (8)$$

Note that (8) does not depend on the number X_j in Π , and thereby not on the N in (2). Therefore, it follows from Lemma 2 and (8) in view of the arbitrariness of X_j and i that for any Π

$$\alpha_{ji-1} - \alpha_{ji} = \alpha > 0, \quad \forall i \in \{2, \dots, m_j\}, \quad \forall X_j \in \Pi.$$

This and (2) give us that f is unique. The second part follows from the fact that in (2) we can set $\alpha_{j1} = m_j - 1$ and $\alpha = 1$.

§4. Some comparisons with PL and BR

It is easy to verify the following two assertions.

COROLLARY 1. *Suppose that f is defined for all Π consisting of strict orderings. Then an f satisfying A0–A2 and A4 is the plurality rule.*

COROLLARY 2. *Suppose that f is defined for all Π consisting of strict orderings. Then an f satisfying A0–A3 and A5 is Borda's rule.*

Note that in the general case BR and the rule determined by Theorem 2 do not coincide. This follows from an example.

EXAMPLE. Let $\Pi = \{X_1, X_2\}$, with

$$X_1 = a \succ b \succ c \succ d,$$

$$X_2 = d \succ b \sim c \succ a.$$

According to the rule defined by Theorem 2, we have that $\alpha_{11} = 3$, $\alpha_{12} = 2$, $\alpha_{13} = 1$, $\alpha_{14} = 0$, $\alpha_{21} = 2$, $\alpha_{22} = 1$, and $\alpha_{23} = 0$. By (2), $f(\Pi) = \{a, b\}$.

According to BR, we have that $\beta_{1i} = \alpha_{1i}$, $i = 1, 2, 3, 4$, $\beta_{21} = 3$, $\beta_{22} = 3/2$, $\beta_{23} = 0$, and by BR

$$f(\Pi) = \left\{ x: \sum_{ji} \beta_{ji}(x) = \max_{g \in X} \sum_{ji} \beta_{ji}(y) = 7/2 \right\} = \{b\}.$$

We make the following remark. Suppose that f is defined for Π consisting of X_j such that $m_j = 2$ or X_j is an indifference (see §2). Then such X_j , or, more precisely, the sets $l_{j1} \subset X$, can be interpreted as a ballot (see [1]). In the same place axioms A0–A2 and one more axiom are used to define a rule according to which $f(\Pi)$ contains the alternatives occurring the greatest number of times in the l_{j1} . But for Π consisting of X_j of the indicated type this rule coincides both with the rule defined by Theorem 1 (axioms A0–A2 and A4) and with the rule defined by Theorem 2 (axioms A0–A3 and A5).

§5. Some remarks on axioms and rules

Since the rules defined by Theorems 1 and 2 are fairly close to PL and BR (the rule characterized by Theorem 1 is a simple generalization of PL), for which axiomatic definitions were known before, it seems appropriate to make some remarks of a bibliographical nature.

Axioms A0 and A1 are fairly transparent. Therefore, it is difficult to say when they were first used to define choice rules and by whom.

Axiom A2 surely appeared first in Smith's paper [8] and almost simultaneously in Young's paper [10].

Axiom A3 was apparently first discussed by Gärdenfors in [3]. As far as I know, it has not been used by anyone to define group choice rules.

Axiom A4 was discussed [4] for decision-making problems with uncertainty. It may first have appeared in [2] for characterizing group choice rules.

Axiom A5 is a greatly weakened modification of axiom A4.

The plurality rule (Theorem 1) was first characterized in [7] only for strict orderings (Corollary 1). We remark that there, in addition to axioms A0–A2 and A4, the author also uses (unnecessarily) the Archimedean axiom (see, for example, [8]).

The rule characterized by Theorem 2, as well as Borda's rule, was apparently first defined axiomatically in [6]. True, it is required there that f be a group ordering rule. The axiomatic characterization in [6] differs from the present change in the separability axiom (A2) by certain conditions on the independence of the alternatives.

Borda's rule was first defined with the help of the separability axiom in [10]. Unfortunately, the conditions characterizing the rule there include one that seems specially chosen just for BR.

The axiomatization of the rule in [1], which can be defined by axioms A0–A2 and A4 or A0–A3 and another variant of A5, also suffers from the specialization of one of the conditions.

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