Towards Structural Tractability in Hedonic Games*

Dominik Peters
Department of Computer Science
University of Oxford, UK
dominik.peters@cs.ox.ac.uk

Abstract

Hedonic games are a well-studied model of coalition formation, in which selfish agents are partitioned into disjoint sets, and agents care about the make-up of the coalition they end up in. The computational problem of finding a stable outcome tends to be computationally intractable, even after severely restricting the types of preferences that agents are allowed to report. We investigate a structural way of achieving tractability, by requiring that agents' preferences interact in a well-behaved manner. Precisely, we show that stable outcomes can be found in linear time for hedonic games that satisfy a notion of bounded treewidth and bounded degree.

Introduction

A *coalition* is an alliance between a group of individuals, formed in order to achieve a common goal. How do such coalitions form if agents are selfish? An extensive literature in economics and computer science has studied this question using the natural model of a *hedonic game* (see the survey by Aziz and Savani (2016)). A hedonic game consists of a set N of agents, each of which submits a (complete, transitive) preference ordering $\succcurlyeq_i$ over all possible coalitions $S \subseteq N$ with $i \in S$. An outcome of the game is a *partition* of the agent set N into disjoint coalitions: agents like a partition π exactly as much as they like their coalition S in π. If agents are selfish, we want to find a *stable* outcome, while in other situations a welfare-optimal or fair outcome might be desired.

Formally, a partition π is *core*-stable if it is resistant to group deviations: there is no non-empty coalition $S \subseteq N$ such that $S \succ_i \pi$ for all $i \in S$. It is *strict-core*-stable if there is no non-empty $S \subseteq N$ such that $S \succcurlyeq_i \pi$ for all $i \in S$, with at least one preference strict. Given a game, we wish to find a partition that is core-stable. To study the computational complexity of this, we will consider the *decision* problem of whether a given hedonic game admits *any* such outcome.

Here, we study what kind of restrictions on the structure of the hedonic games involved can be used to make this problem tractable, and argue that restrictions on the *entire preference profile* rather than individual preferences are most promising.

*This is a 'student abstract' based on material in Peters (2016a; 2016b; 2015). I wish to thank Edith Elkind for advice and guidance.

Finding Friends is Hard

Annoyingly, the problem of finding a core-stable partition in a hedonic game as just described has turned out to be computationally hard in almost all cases. In order to make this claim precise, we need a way to succinctly represent agents' preferences over the exponential space of all coalitions. An influential and universally expressive way to do this are *hedonic coalition nets* (or *HC-nets*), introduced by Elkind and Wooldridge (2009), in which agents are characterised by a list of weighted boolean *rules*: for example, the rule $b \land \neg c \mapsto_a 3$ means that agent a derives utility 3 in coalitions that include agent b but not c. The total utility of a in a coalition S is the sum of the weights of all rules satisfied in S.

As Elkind and Wooldridge (2009) show, deciding whether a game given by an HC-net admits a core-stable outcome is Σ_2^p-complete. This comes as a blow to the project of finding stable outcomes: the problem encapsulates the full hardness of the second level of the polynomial hierarchy, and is thus considerably harder than 'merely' NP-complete problems.

What can be done? A natural and initially promising approach was to *restrict agents' preferences*: what if, for example, agents' preferences are *additively separable*? Here, agents assign other agents numbers $v_i(j)$ and evaluate coalitions according to the sum of the values in it: $\sum_{j \in S} v_i(j)$. Alas, this does not make finding stable solutions easier:

Theorem 1 (Peters 2015)**.** *The problems of deciding whether a given additively separable hedonic game admits a core- or a strict-core-stable partition are Σ_2^p-complete, even if valuations are symmetric ($v_i(j) = v_j(i)$) and sparse: no agent has non-zero valuations for more than 10 agents.*

This result is an extension of a result of Woeginger (2013). An impressive number of other preference restrictions have been studied, and invariably NP-hardness remains: in particular, this is the case when agents only care about their most or least preferred agent in a coalition (Hajduková 2006), or if agents take the *average* of the $v_i(j)$ rather than the sum (Brandl, Brandt, and Strobel 2015), and Σ_2^p-hardness remains for the strict core even if preferences are *dichotomous* (Peters 2015). Indeed, finding a core-stable outcome must remain hard for all (succinct) preference representations which allow agents a minimum of expressive freedom, in that they are allowed to arbitrarily rank coalitions of form $\{i, j\}$ and they are allowed to dislike some players (Peters and Elkind 2015).

The problem usually remains hard even after imposing further conditions such as symmetry, strictness, or sparseness.

Structural Tractability: Bounded Treewidth

Despite some isolated positive results – some examples for dichotomous preferences can be found in Peters (2016a) and are summarised in Table 1 – it seems fair to conclude that the approach of identifying restrictions on individual agents' preferences to allow tractability has substantially failed. An alternative method of attack is needed, and here we propose a *structural* one. The idea is to consider subclasses of hedonic games in which agents' preferences are *correlated* in that they, taken *together*, appear structured. As a somewhat trivial example, consider the *common ranking property* of Farrell and Scotchmer (1988), where every agent's preference agrees with a common ranking of all coalitions. As can be easily seen, successively picking top coalitions according to the common ranking yields a core-stable outcome.

More interestingly, let us consider the idea that *cyclicity* can be viewed as a main source of hardness, so that minimising the occurrences of cyclic relationships in a problem might make it easier. (Indeed, it is notable that essentially all examples of hedonic games that do not admit stable outcomes depend on an odd number of agents placed in a cycle.) A particularly successful way of limiting cycles is through bounding the *treewidth* of an underlying graph describing the problem instance. The treewidth of a graph is a measure of how 'tree-like' a given graph is – for definitions and an overview we refer the reader to Bodlaender (1994).

In particular, for a game given by an HC-net, we can define a natural *dependency graph* G with vertex set N the agents, where an edge connects i and j if j appears in a rule of agent i. Thus, agents that care about each other's presence are connected in G. Elkind and Wooldridge (2009) asked whether finding core-stable outcomes becomes tractable restricted to games whose dependency graph has bounded treewidth. As we show, the answer in general is *no*, since deciding the existence of a core-stable partition remains hard for games of treewidth 2 (Peters 2016b). However, tractability arises once we additionally bound the maximum *degree* of G. This tractability is not restricted to just core-stability: a wide range of other computational questions can be efficiently answered on such a domain, allowing us to find partitions that are, for example, envy-free or Pareto-optimal.

Theorem 2 (Peters 2016b)**.** *For any logical sentence ϕ of* HG-logic *that may quantify over (connected) partitions, coalitions, and agents, the problem of deciding whether a hedonic game given by an HC-net satisfies ϕ is fixed-parameter tractable with parameters the treewidth and degree of the dependency graph of the game.*

Proof idea. Encode HG-logic into monadic second-order logic, and use Courcelle's theorem. □

As an example, $\phi \equiv \exists \pi \, \forall S \, \exists i \;\; (i \in S \wedge \pi \succcurlyeq_i S)$ encodes the existence of a core-stable partition. This result is encouraging in that it applies to many concepts ϕ and does not depend on severe restrictions on individual agents' preferences who may use any HC-net subject to the degree bound.

	SW	PF	PO	NS	IS	CR	SCR
Boolean	NP-c.	NP-c.	NP-h.	NP-c.	P	FNP-h.	Σ_2^p-c.
2-lists	NP-c.	P	P	P	P	P	NP-c.
3-lists	NP-c.	NP-c.	NP-h.	?	P	P	NP-c.
4-lists	NP-c.	NP-c.	NP-h.	NP-c.	P	P	NP-c.
Anonymous	NP-c.	NP-c.	NP-h.	NP-c.	P	P	NP-c.
Intervals	P	P	P	?	P	P	?
Roommates	P	P	P	NP-c.	P	P	P
Majority	?	P	?	P	P	P	P

Table 1: Complexity results for various dichotomous preference representations, see Peters (2016a) for details.

Future Work

The notion of structural tractability suggests multiple avenues for potentially fruitful future work. As an example, suppose agents are ordered along a line, and have additively separable preferences that are *single-peaked* on that line: agents prefer those agents closer to them on the line. Does every such game admit a core-stable outcome? If so, can we find it in polynomial time? Similarly, what happens if all individually rational coalitions are *intervals* with respect to this line? Some positive results for the dichotomous case can be found in Peters (2016a). In other possible models, agents may be points in a metric space, and may prefer to be relatively 'central' in their coalition.

More directly related to the results presented here, we may ask whether runtime improvements are possible for preferences of bounded treewidth that are additionally restricted in some 'traditional' way. In particular, can we dispense with the bound on G's maximum degree when only considering additively separable games?

References

Aziz, H., and Savani, R. 2016. Hedonic games. In *Handbook of Computational Social Choice*. Cambridge University Press.

Bodlaender, H. L. 1994. A tourist guide through treewidth. *Acta cybernetica* 11(1-2):1.

Brandl, F.; Brandt, F.; and Strobel, M. 2015. Fractional Hedonic Games: Individual and Group Stability. AAMAS '15, 1219–1227.

Elkind, E., and Wooldridge, M. 2009. Hedonic coalition nets. AAMAS '09, 417–424.

Farrell, J., and Scotchmer, S. 1988. Partnerships. *The Quarterly Journal of Economics* 279–297.

Hajduková, J. 2006. Coalition Formation Games: A Survey. *International Game Theory Review (IGTR)* 08(04):613–641.

Peters, D., and Elkind, E. 2015. Simple causes of complexity in hedonic games. IJCAI '15, 617–623, arXiv:1507.03474 [cs.GT].

Peters, D. 2015. Σ_2^p-complete problems on hedonic games. arXiv:1509.02333 [cs.GT].

Peters, D. 2016a. Complexity of hedonic games with dichotomous preferences. AAAI '16.

Peters, D. 2016b. Graphical hedonic games of bounded treewidth. AAAI '16.

Woeginger, G. J. 2013. A hardness result for core stability in additive hedonic games. *Mathematical Social Sciences* 65(2):101–104.