Single-Peakedness and Total Unimodularity for Multiwinner Elections

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Multiwinner Elections

- **Input:** Finite set $N$ of agents and set $C$ of candidates; for each agent $i \in N$ a preference relation $\succeq_i$ over $C$.
- **Output:** A *committee* $W \subseteq C$ of exactly $k$ candidates.
- So need a voting rule $f : \mathcal{R}(C)^N \to \{W \subseteq C : |W| = k\}$.
- **Applications:** Proportional parliaments, representative committees, shortlisting, movies on airplanes, . . .
- **Problem:** Many popular rules optimise an objective function over all $\binom{m}{k}$ committees $W \rightsquigarrow$ NP-hard winner determination.
Structured Preferences

- One way to deal with hardness: look for tractable classes of profiles.
- Various forms of *structure* are promising.
- Main example: *single-peaked* preferences.
- Alternative space is one-dimensional, e.g. numerical quantities.
Single-Peaked Preferences

- Suppose alternative space admits a left-to-right ordering $\prec$.
- Then $\succeq_i$ is single-peaked w.r.t. $\prec$ if whenever $\text{top}(i) \prec a \prec b$ or $b \prec a \prec \text{top}(i)$, then $a \succeq_i b$.

- A profile is single-peaked if there is some $\prec$ such that each $\succeq_i$ is single-peaked w.r.t. $\prec$.
- **Fact:** Can decide efficiently whether a profile is single-peaked.
One popular multiwinner voting rule is due to political scientists Chamberlin and Courant (1983)

**Idea:** Every voter is *represented* by their favourite candidate in \( W \), and obtains corresponding utility

\( \rightsquigarrow \) return committee with highest utilitarian welfare

Committee \( W \subseteq C \) gets objective value

\[
\sum_{i \in N} \max \{ \text{Borda-score}_i(c) : c \in W \}.
\]

NP-complete to find optimum committee.
Betzler, Slinko, Uhlmann (JAIR 2013): CC is easy to compute when preferences are single-peaked.

Dynamic Programming algorithm:
1. Find axis witnessing single-peakedness.
2. Build up committee from left to right.
3. In DP table, compute how much we gain by adding \( c \) to committee, by tracking who would be represented by \( c \).
Proportional Approval Voting

- **Criticism of CC:** Only gain utility from one committee member, ignore rest.

- Alternative rule: PAV, working with approval ballots (dichotomous preferences)

- **Idea:** voters gain utility from each committee member that they approve, but *decreasing marginal returns*.

- PAV assigns committee \( W \subseteq C \) the objective value

\[
\sum_{i \in N} \left( 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{|W \cap v_i|} \right).
\]

- NP-complete to find optimum committee.

- Can use non-harmonic weights, but these give good *proportionality* properties (Aziz et al., SCW 2016).
For approval ballots, single-peaked means “approve an interval”

Elkind and Lackner (IJCAI 2015) study whether PAV becomes easy for single-peaked preferences

Problem: The Betzler et al. algorithm does not work for PAV – voters can have many “representatives”

The Betzler et al. algorithm works if voters only approve few candidates – then there are few representatives (Elkind and Lackner 2015)

Elkind and Lackner also find tractability for some further restricted subclasses (e.g., when every voter approves one of the extreme candidates)

Elkind and Lackner conjecture: PAV remains NP-hard for general single-peaked preferences.
PAV: Integer Programming Formulation

maximise \[ \sum_{i \in N} \sum_{\ell \in [k]} \frac{1}{\ell} \cdot x_{i,\ell} \]  

(subject to \(9/16\))

\[ \sum_{c \in C} y_c = k \quad (2) \]

\[ \sum_{\ell \in [k]} x_{i,\ell} \leq \sum_{i \text{ approves } c} y_c \quad \text{for } i \in N \quad (3) \]

\[ x_{i,\ell} \in \{0, 1\} \quad \text{for } i \in N, \ \ell \in [k] \quad (4) \]

\[ y_c \in \{0, 1\} \quad \text{for } c \in C \]
Total Unimodularity

- A \{-1, 0, 1\}-matrix \(A\) is called **totally unimodular** (TU) if for every square submatrix \(B\) of \(A\) has \(\det B \in \{-1, 0, 1\}\).
- If \(A\) is TU, then the integer program
  \[
  \max c^T x \text{ subject to } Ax \leq b, x \in \mathbb{Z}^n
  \]
  (IP)
  is solved optimally by its LP relaxation \(\rightsquigarrow\) poly-time solvable.
- Every matrix with the **consecutive ones property** is TU.

\[
\begin{bmatrix}
0 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
\end{bmatrix}
\]
Theorem: If preferences are single-peaked, then (PAV-ILP) is TU and hence poly-time solvable.

Example: 4 candidates $a, b, c, d$, committee size $k = 2$
Voter 1 approves $\{a, b, c\}$, and Voter 2 approves $\{c, d\}$.
Single-peaked: $a \prec b \prec c \prec d$.

$$\text{maximise } (x_{1,1} + \frac{1}{2}x_{1,2}) + (x_{2,1} + \frac{1}{2}x_{2,2}) \quad (\text{PAV-IP'})$$
subject to
$$y_a + y_b + y_c + y_d = 2 \quad (2)$$
$$x_{1,1} + x_{1,2} \leq y_a + y_b + y_c \quad (3)$$
$$x_{2,1} + x_{2,2} \leq y_c + y_d \quad (3)$$

all variables binary

$$A_{\text{PAV-IP'}} = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
-1 & -1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & -1 & -1 & 0 & 0 & 1 & 1
\end{bmatrix}$$
Other rules

- For CC, one can also design an ILP formulation that becomes TU if preferences are single-peaked.

\[
\text{maximise } \sum_{i \in N} \sum_{r \in [m]} x_{i,r} \\
\text{subject to } \sum_{c \in C} y_c = k \\
x_{i,r} \leq \sum_{c : \text{rank}(c) \geq r} y_c \quad i \in N, \ r \in [m] \\
x_{i,r}, y_c \in \{0, 1\}
\]

- This gives a poly-time algorithm for the single-peaked case, but it is not a specialised algorithm: it works also if the input is not single-peaked.

- This algorithm does not first need to find an axis \( \triangleleft \).

- Note that the input preferences are encoded in the constraints, not the objective function (as would be more natural).
The same approach works for generalisations of CC via ordered weighted averages: *OWA-based rules* (Skowron et al. AIJ 2016).

For example, this includes the “t-Borda rules”, in which voters are represented by their $t$ favourite committee members.

Also works for egalitarian variants of CC and PAV.
Suppose alternatives are arranged in a *circle*.
Preferences are single-peaked on this circle (SPOC) if for every voter, we can cut the circle so that the vote is single-peaked on the resulting line (Peters and Lackner, AAAI 2017).

The TU approach can also be made to work for SPOC profiles: CC, PAV, etc. remain tractable for this larger domain.
Conclusions and Future Work

- New method of designing algorithms for single-peaked settings
- Are there other problems for which this works?
- Open: PAV with “voter interval” preferences (Elkind and Lackner 2015)
- Open: Characterise class of profiles for which the ILPs shown are TU.
- Approach allows to optimise any linear objective
  ↦ application to certain facility location problems on a line.
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