Preferences Single-Peaked on a Circle*

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We introduce the domain of preferences that are single-peaked on a circle, which is a generalization of the well-studied single-peaked domain. This preference restriction is useful, e.g., for scheduling decisions, certain facility location problems, and for one-dimensional decisions in the presence of extremist preferences. We give a fast recognition algorithm of this domain, provide a characterisation by finitely many forbidden subprofiles, and show that many popular single- and multi-winner voting rules are polynomial-time computable on this domain. In particular, we prove that Proportional Approval Voting can be computed in polynomial time for profiles that are single-peaked on a circle. In contrast, Kemeny's rule remains hard to evaluate, and several impossibility results from social choice theory can be proved using only profiles in this domain.

1. Introduction

A central problem in the study of multi-agent systems is the aggregation of agents' preferences in order to make group decisions. Impossibility theorems and computational hardness result make this problem a hard one to solve. However, a successful line of research going back to Black's [1948] seminal article has managed to circumvent many problems in (computational) social choice for the special case when agents' preferences are *single-peaked*. Under this preference restriction, we assume that agents have preferences over the possible values of a one-dimensional quantity such as the timing of a deadline, a tax rate, a thermostat setting, or the price of a new product. A preference ordering is *single-peaked* if an agent has a certain most-preferred value of the quantity and derives less and less satisfaction from values that are further away from the subjective optimum. Another popular application of this setting is in political elections, where it is often held that candidates can be ordered on a left-to-right spectrum making the voters' preferences single-peaked.

Preference profiles that consist solely of single-peaked preference orderings have attractive properties, both algorithmically and in terms of their social choice behaviour [Elkind et al., 2016, 2017].

*This paper is based on the two conference publications *Preferences Single-Peaked on a Circle* [Peters and Lackner, 2017] and *Single-Peakedness and Total Unimodularity: New Polynomial-Time Algorithms for Multi-Winner Elections* [Peters, 2018].

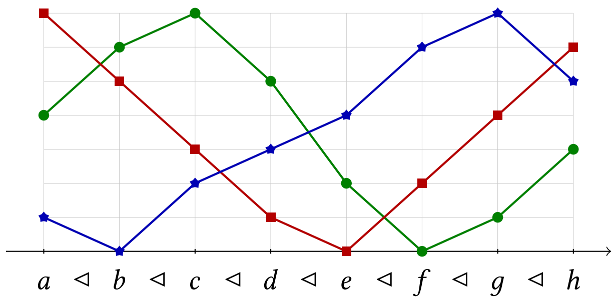


Figure 1: Example of preferences single-peaked on a circle.

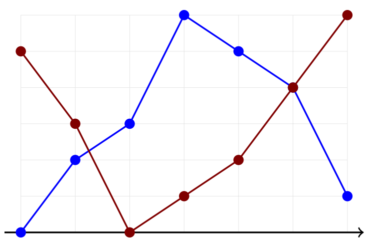
For example, winner determination problems that are computationally hard in the general case tend to be easy when restricted to single-peaked profiles [Betzler et al., 2013, Brandt et al., 2015], and the single-peaked domain guarantees the existence of Condorcet winners as well as transitivity of the majority relation and thus admits a strategyproof voting rule [Moulin, 1991].

The usefulness of results of this type is limited by the extent to which profiles in practice actually happen to be single-peaked. One way of dealing with this is to consider less restrictive generalisations of single-peakedness. Maybe the structure of the alternative space is not quite one-dimensional, and in this case it might be useful to consider preferences that are single-peaked on a *tree* [Demange, 1982]. This domain is notably larger, yet still retains many desirable properties in social choice terms; however, its algorithmic usefulness is more mixed [Yu et al., 2013, Peters and Elkind, 2016].

In this paper, we identify a new preference restriction: being single-peaked on a *circle*. Here we assume that alternatives can be placed on a circle, with agents' preferences again being decreasing on both sides of their peaks. See Figure 1 for some example shapes that 'preference curves' might have in this setting; higher points are more preferred. Note that the circle wraps around, and so h and a are adjacent. Intuitively, a preference profile is single-peaked on a circle if, for every agent, we can 'cut' the cycle once so that the agent's preferences are single-peaked on the resulting line. Crucially, the location of the cutting point may differ for each agent.

The aim of this paper is to explore this new preference domain in detail. We will find that this domain is algorithmically useful (it often allows for efficient winner determination), but it performs less convincingly in terms of axiomatic properties (since voting paradoxes still occur and impossibility results can still be proven). Interestingly, this is precisely opposite to how the results turned out for single-peakedness on trees.

1.1. Motivating Examples



There are many practical scenarios where we might expect preferences to be single-peaked on a circle. This is even the case when, on first sight, there seems to be no circle anywhere. Indeed, suppose that alternatives are naturally ordered on a line; we may pretend this line is a circle by joining up its endpoints. Of course, every order that is single-peaked on the line is also single-peaked on the circle. But crucially, the *reverse* of such an order, now single-*caved* on the line, is still single-peaked on the same circle. Thus, our new preference restriction allows us to *combine* single-peaked and single-caved votes (as shown on the right). One interpretation is that this move allows us to accommodate "extremists". For example, while most people have a sweet spot somewhere on the left-right political axis, some people might dislike centrist options and prefer the extremes. When planning a vacation, some might have an optimal climate in mind, while others like it both very cold (skiing) and very hot (beaches), but dislike compromises (England).

Other examples of alternative spaces are more explicitly cyclic. Consider, for example, finding a good time for a daily event (such as a day or night shift, or a meeting, or the timing of backups) where possibilities are arranged in a 24-hour cycle. A similar structure exists when scheduling an international phone call; here, different time zones are arranged along the equator, and lead to cyclic preferences.

But perhaps the most appealing example of preferences that can be expected to be single-peaked on a circle come from problems inspired by facility location. Rather many structures have a boundary that is (roughly) isomorphic to a cycle, including most cities and countries. The problem of deciding where to locate a new airport for a city would be one example, since airports are usually positioned on the boundary. Similarly, where should a company build new office space? To which coastal region should a family move? Where do we want to sit in a football stadium? Another plausible application could be inspired by security concerns, if we consider the placement of border security checkpoints.

1.2. Contributions

The main contributions of this paper can be summarized as follows:

- We formally define single-peakedness on circles and immediately extend this definition to preferences with ties, and to dichotomous (approval) preferences. Thus, our proposed domain is strictly larger than the class of *possibly single-peaked preferences* [Lackner, 2014] and *candidate interval* dichotomous preferences [Faliszewski et al., 2011, Elkind and Lackner, 2015].

- We show that it is possible to efficiently recognise whether a given preference profile is single-peaked on some circle, and if so return a suitable circle. For the case of preferences without ties, we give a recognition algorithm that runs in linear time, matching the performance in the case for the line.

- We characterise the domain of preferences single-peaked on a circle through a list of finitely many *forbidden subprofiles* with 2 voters and 5 alternatives, and with 3 voters and 4 alternatives. The proof of this characterisation implies that our linear-time recognition algorithm can *certify* its negative answers by exhibiting a forbidden subprofile.

- While single-peakedness on a line serves as a way to circumvent many impossibility results in social choice, we show that such impossibilities (including the famous impossibility result by Gibbard and Satterthwaite) can still be proven when preferences are allowed to be single-peaked on a circle.

- We then study the algorithmic properties of our new preference extension. We show that Young's voting rule (and also Young *scores*) can be efficiently computed if preferences are single-peaked on a circle; this algorithm also improves upon the state-of-the-art when it comes to preferences single-peaked on a line. In contrast, we show that Kemeny's rule is NP-hard to compute even in this restricted domain.

- Finally, we show that several multi-winner voting rules are efficiently computable in our restricted case, specifically all that are included in the large class of so-called OWA-based rules. This class includes, e.g., the *Chamberlin–Courant* rule and *Proportional Approval Voting* (PAV). It is noteworthy that some of these algorithmic results have not yet been established even for single-peaked profiles (such as the one for PAV). This general result relies on using total unimodularity and integer programming.

1.3. Outline of the Paper

In Section 2, we introduce and define single-peakedness on a circle. In Section 3, we discuss the algorithmic recognition of these preferences; proof details are delegated to the appendix, Section A. Building on these results, we prove a characterization of this preference class in Section 4, with proof details in the appendix, Section B. We then take the perspective of social choice theory in Section 5 and revisit classical impossibility results. The algorithmic usefulness of preferences that are single-peaked on a circle is the focus of the following two sections: Section 6 for the single-winner rules Kemeny and Young, and Section 7 for several multi-winner rules including Chamberlin–Courant and Proportional Approval Voting. Open problems are discussed in Section 8.

2. Definition

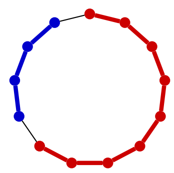
Let A be a finite set of *alternatives* (or *candidates*). A *weak order* (or *preference relation*) is a binary relation $\succcurlyeq$ over A which is complete and transitive. A *linear order* is a weak order that is antisymmetric, and so does not allow preference ties; a *strict* linear order $\succ$ is the irreflexive part of a linear order. A *profile* $P = (v_1, \dots, v_n)$ over A is a list of weak orders over A. The elements of $N = \{1, \dots, n\}$ are called *voters*, and we associate voter $i \in N$ with the order v_i, which we call the *vote* of voter i. For convenience, we write $a \succcurlyeq_i b$ whenever $(a, b) \in v_i$, i.e., when voter i weakly prefers alternative a to alternative b. We also use $\succ_i$ and $\sim_i$ for the strict and indifference parts of $\succcurlyeq_i$. We will always write m for the number of alternatives and n for the number of voters. If v_i is a linear order, we write $\text{top}(v_i)$ for i's most-preferred alternative.

An *axis* $\lhd$ is a strict linear order of the alternatives. We usually think of an axis as describing the underlying one-dimensional structure of the alternative space. A linear order v_i is *single-peaked* with respect to the axis $\lhd$ if for each pair of alternatives $a, b \in A$ with $\text{top}(v_i) \lhd b \lhd a$ or $a \lhd b \lhd \text{top}(v_i)$ it holds that $b \succ_i a$. Let us also give another, equivalent definition. An *interval* $I \subseteq A$ of an axis $\lhd$ is any set such that for all $a, b, c \in A$, if we have $a, c \in I$ and $a \lhd b \lhd c$, then $b \in I$. Then a vote v_i is single-peaked with respect to the axis $\lhd$ if and only if for every $c \in A$, the top-initial segment $\{a \in A : a \succ_i c\}$ is an interval of $\lhd$. This definition in terms of intervals immediately gives a definition of the single-peaked property for *weak* orders as well. There are several possible definitions of single-peakedness for weak orders; the one stated above is sometimes known as *possible single-peakedness* [Lackner, 2014], since it is equivalent to saying that there exists a linearisation of the weak order which is single-peaked.

We say that two axes $\lhd$ and $\lhd'$ are *cyclically equivalent* if there is $l \in [m]$ such that we can write $a_1 \lhd a_2 \lhd a_3 \lhd \cdots \lhd a_m$ and $a_l \lhd' a_{l+1} \lhd' \cdots \lhd' a_m \lhd' a_1 \lhd' \cdots \lhd' a_{l-1}$. For an axis $\lhd$, we then define the *circle* $C(\lhd)$ *of* $\lhd$ to be the set of axes cyclically equivalent to $\lhd$. Any set C of axes that can be written as $C = C(\lhd)$ for some $\lhd$ we call a *circle*. For example, $C = \{a \lhd b \lhd c, b \lhd' c \lhd' a, c \lhd'' a \lhd'' b\}$ is a circle. Note that "cutting" a circle C at a point yields an axis $\lhd \in C$. We say that $\lhd$ *starts in* $a \in A$ if $a \lhd b$ for all $b \in A \setminus \{a\}$.

Definition. *Let* C *be a circle. A vote* v_i *is* single-peaked on C *if there is an axis* $\lhd \in C$ *such that* v_i *is single-peaked with respect to* $\lhd$. *A preference profile* P *is* single-peaked on a circle (SPOC) *if there exists a circle* C *such that every vote* $v_i \in P$ *is single-peaked on* C.

Intuitively, a vote v_i is single-peaked on C if C can be cut so that v_i is single-peaked on the resulting line.



Again let us state another equivalent definition. An *interval* $I \subseteq A$ of a circle C is a set that is an interval of one of the axes $\lhd \in C$ of the circle. Then a vote is single-peaked on a circle C if and only if each top-initial segment $\{a \in A : a \succ_i c\}$ is an interval of C. Note that the complement $A \setminus I$ of an interval I of C is again an interval. Thus, a weak order $\succcurlyeq$ is single-peaked on C if and only if its reverse $\overleftarrow{\succcurlyeq} = \{(b, a) : (a, b) \in \succcurlyeq\}$ is also single-peaked on C.

A vote is *single-caved* if its reverse is single-peaked. It follows, then, that mixtures of single-peaked and single-caved orders (on the same axis) are SPOC. However, not all SPOC profiles have this form; one such example is the profile shown in Figure 1, where the circle cannot be cut so as to make every preference curve either single-peaked or single-caved.

A weak order $\succcurlyeq$ is *dichotomous* if there is a partition of A into sets A_1 and A_2 such that $a \succ b$ if and only if $a \in A_1$ and $b \in A_2$. A voter whose preferences are given by $\succcurlyeq$ is said to *approve* the alternatives in A_1. Note that, according to our definition, a profile of dichotomous (approval) preferences is SPOC if and only if there is a circle C such that every voter's approval set is an interval of C.

3. Recognition Algorithms

In this section we design algorithms that decide whether a given profile is single-peaked on some circle, and if so, return a suitable circle C.

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

A matrix $M = (a_{ij})$ with $a_{ij} \in \{0, 1\}$ has the *consecutive ones property* if the columns of M can be put into a linear order $\lhd$ so that for every row of M, the columns with 1-entries form an interval of $\lhd$. The matrix shown on the right is an example. The matrix M has the *circular ones property* if its columns can be arranged in a circle C so that the 1-entries of each row form an interval of C. Given our definitions in terms of intervals above, it is straightforward to translate a profile P of weak orders into an $mn \times m$ matrix M so that P is single-peaked [single-peaked on a circle] if and only if M has the consecutive [circular] ones property [Bartholdi III and Trick, 1986]: Take one column for each alternative, and one row for every top-initial segment of every voter in P; the row is the incidence vector of the segment. Since it is possible to check in linear time whether a matrix A has the consecutive or circular ones property [Booth and Lueker, 1976], this gives us an $O(m^2 n)$ algorithm to recognise profiles that are single-peaked on a circle.

In the remainder of this section, we design a more explicit algorithm that runs in time $O(mn)$ when the input profile consists of *linear* orders.[1] The algorithm works by reducing the SPOC recognition

[1] Actually, the algorithm works whenever P contains at least one linear order.

problem for linear orders to the recognition problem of single-peaked profiles for weak orders, in such a way that we can apply an algorithm of Fitzsimmons and Lackner [2019].

Suppose $P = (v_1, \dots, v_n)$ is a profile of linear orders over A, and fix some alternative $z \in A$. We will build another profile $\bar{P} = (v_1^u, v_1^l, \dots, v_n^u, v_n^l)$ of $2n$ *weak* orders by *slicing* each vote v_i *at* z into an *upper* part v_i^u and a *lower* part v_i^l. The upper part v_i^u ranks all alternatives a such that $a \succ_i z$ in order of $\succ_i$, and puts all remaining alternatives into a least-preferred indifference class. The lower part v_i^l ranks all alternatives a such that $z \succ_i a$ in *reverse* order of $\succ_i$, and again puts all remaining alternatives into a least-preferred indifference class.

Example. *Slicing the order* $a \succ b \succ c \succ z \succ d \succ e \succ f$ *at* z *yields the upper part* $a \succ^u b \succ^u c \succ^u z \sim^u d \sim^u e \sim^u f$ *and the lower part* $f \succ^l e \succ^l d \succ^l z \sim^l a \sim^l b \sim^l c$.

The notion of slicing reduces SPOC to single-peakedness:

Proposition 1. *Suppose a profile* $\bar{P}$ *of weak orders is obtained by slicing each vote of a profile* P *of linear orders at some fixed* $z \in A$. *Then* P *is SPOC if and only if the profile* $\bar{P}$ *is single-peaked.*

Proof. Suppose P is SPOC on C, and let $\lhd \in C$ be an axis starting in z. Since z is least-preferred by all voters in $\bar{P}$, z is not contained in any top-initial segment of any voter in $\bar{P}$. However, all top-initial segments of votes in $\bar{P}$ are intervals of C. Since they do not contain z, they must also be intervals of $\lhd$. Thus, $\bar{P}$ is single-peaked with respect to $\lhd$.

Suppose $\bar{P}$ is single-peaked with respect to $\lhd$. We show that P is SPOC on $C = C(\lhd)$. Take a top-initial segment S of a vote v_i in P; we prove that S is an interval of C. If $z \notin S$, then S is a top-initial segment of v_i^u in $\bar{P}$. Thus, S is an interval of $\lhd$ and so an interval of C. If however $z \in S$, then the complement $A \setminus S$ is a top-initial segment of v_i^l in $\bar{P}$, hence an interval of $\lhd$, and so $A \setminus S$ is an interval of C. But the complement of an interval of a circle is again an interval, and so S is an interval of C. Hence P is SPOC. ◻

Thus, we can use an algorithm that decides whether a profile of weak orders is single-peaked to decide whether a profile of linear orders is SPOC. Next, note that if we select $z \in A$ to be the alternative that is ranked last by v_1 (say), then the profile $\bar{P}$ obtained by slicing P at z contains a linear order (namely the upper part of v_1). Fitzsimmons and Lackner [2019] give an $O(mn)$ time algorithm that decides whether a profile of weak orders containing at least one linear order is single-peaked. In Appendix A, for completeness, we include a description of the algorithm of Fitzsimmons and Lackner [2019], and the relevant parts of its correctnes proof.

Since $\bar{P}$ can be constructed from P in time $O(mn)$, by Proposition 1, we obtain the following.

Theorem 2. *There is an* $O(mn)$ *time algorithm that decides whether a profile of linear orders is single-peaked on a circle.*

4. Characterisation by Forbidden Subprofiles

Ballester and Haeringer [2011] have characterised the domain of single-peaked profiles of linear orders by a finite collection of forbidden subprofiles. More precisely, they gave forbidden profiles with 3 voters and 3 alternatives, and with 2 voters and 4 alternatives such that a profile P is *not* single-peaked if and only if it is possible to obtain one of their forbidden profiles from P by deleting and reordering voters, and deleting and renaming alternatives. A similar characterisation exists for *single-crossing* profiles [Bredereck et al., 2013], but no finite characterisation exists for d*-Euclidean* profiles [Chen et al., 2017, Peters, 2017].

Here, we prove that a profile is SPOC unless it contains certain forbidden subprofiles with 2 voters and 5 alternatives and with 3 voters and 4 alternatives. For sets $B, C \subseteq A$ of alternatives, let us write $B \succ_i C$ to mean that $b \succ_i c$ for all $b \in C$ and $c \in C$.

Theorem 3. *A profile P of linear orders on A is* not *SPOC if and only if one of the following three cases occurs.*

1. *There are distinct alternatives $a, b, c, d, e \in A$ and voters v_i and v_j in P such that*

$$\{a, b\} \succ_i \{c\} \succ_i \{d, e\},$$
$$\{a, e\} \succ_j \{c\} \succ_j \{d, b\}.$$

2. *There are distinct alternatives $a, b, c, d \in A$ and voters v_i, v_j, and v_k in P such that*

$$\{a, b\} \succ_i \{c, d\},$$
$$\{a, c\} \succ_j \{b, d\},$$
$$\{a, d\} \succ_k \{b, c\}.$$

3. *There are distinct alternatives $a, b, c, d \in A$ and voters v_i, v_j, and v_k in P such that*

$$\{a, b\} \succ_i \{c, d\},$$
$$\{b, c\} \succ_j \{a, d\},$$
$$\{c, a\} \succ_k \{b, d\}.$$

Proof. Sufficiency. We prove that if one of the three cases occurs, then P is not SPOC. Since SPOC is closed under alternative deletion, in each case, we may assume wlog that P only involves alternatives mentioned in the forbidden condition. Suppose P was single-peaked on the circle C.

1. Considering top-initial segments of size 2, we see that a must have neighbours b and e in C. Considering top-initial segments of size 3, and taking complements, we see that d must have neighbours b and e in C. But this uniquely determines a circle with $a \lhd b \lhd d \lhd e \lhd a$; yet this circle does not include c, a contradiction.

2. Considering top-initial segments of size 2, we see that a must have neighbours b, c, and d in C. But no vertex of a circle has three neighbours, a contradiction.

3. Since P is single-peaked on C, so is P with every order reversed. But after reversing every order, we are again in case 2: d must have three neighbours.

Necessity. This direction is much more involved, and the full proof appears in Appendix B. The proof strategy is as follows: if P is not SPOC, then the recognition algorithm of Theorem 2, run on input P, will return a negative answer. We analyse every way the algorithm could answer negatively, and in each case construct a witnessing forbidden structure from among those identified in the theorem statement. □

For the benefit of future research, let us briefly describe how we obtained Theorem 3. We first implemented a recognition algorithm for SPOC profiles (Theorem 2) and then iterated through all possible profiles of certain sizes (up to isomorphism), checking for each whether they were *minimal counterexamples*: not SPOC, but every profile obtained by deleting a voter or an alternative is SPOC. We analysed the resulting list by hand to come up with the compact representation in Theorem 3. The proof strategy of case analysis of the "no"-conditions of a recognition algorithm is (implicitly) also the approach used in previous characterisations [Ballester and Haeringer, 2011, Bredereck et al., 2013, Cornaz et al., 2012].

5. Impossibility Theorems

One of the major advantages of the traditional single-peaked domain is the existence of a non-manipulable voting rule on this domain: The well-known median voter procedure sorts voters' most preferred alternatives according to the axis $\lhd$ and then returns the median alternative a. This alternative is, in fact, a (weak) Condorcet winner: for any other alternative b, a (weak) majority of voters prefers a to b. One might hope to be able to extend this procedure to circles, but this turns out to be impossible: the Gibbard–Satterthwaite theorem can be proven using only profiles that are single-peaked on a circle.

A *resolute voting rule* f on SPOC profiles is a function assigning a single winning alternative to every SPOC profile of linear orders. The rule f is *non-dictatorial* if there is no fixed voter i such that f always picks i's top alternative. The profile obtained from P by replacing vote v_i by v_i' is denoted by (P_{-i}, v_i'). A voting rule f on SPOC profiles is *strategyproof* if $f(P) \succcurlyeq_i f(P_{-i}, v_i')$ for all orders v_i' such that (P_{-i}, v_i') is still SPOC.

Theorem 4 (Gibbard–Satterthwaite Theorem for SPOC)**.** *There is no resolute voting rule on SPOC profiles that is non-dictatorial, onto, and satisfies strategyproofness.*

Proof. This follows immediately from the results of Kim and Roush [1980] and Sato [2010], who prove this result for an even more restricted domain consisting only of the $2m$ orders which traverse the circle clockwise and counter-clockwise starting from every possible alternative. □

Note that the SPOC orders used in this proof are 'unbalanced', in that the most- and least-preferred alternatives are adjacent on the circle for every agent. Still, a similar dictatorship result can be proved even using orders that are 'Euclidean' on a circle, where preferences decrease uniformly in both directions from the peak [Schummer and Vohra, 2002]. It can also be shown that, with these Euclidean orders, the *random dictatorship* rule is *group*-strategyproof [Alon et al., 2010b], and there is an intriguing randomized mechanism that is strategyproof and provides a 3/2-approximation to the egalitarian social welfare [Alon et al., 2010a].

Another desirable axiomatic property is *participation*, which, intuitively, states that no voter can strictly benefit by abstaining from an election. Formally, f satisfies participation if for all profiles P, and all linear orders $\succ$, we have $f(P + \succ) \succcurlyeq f(P)$, where $P + \succ$ denotes the profile obtained from P by adding a voter with preferences given by $\succ$. A celebrated result of Moulin [1988] shows that this property is incompatible with *Condorcet-consistency*, which requires that if in a profile P, there exists a Condorcet winner a, then $f(P) = a$. Moulin's result can be proven using only SPOC profiles.

Theorem 5 (No-Show Paradox for SPOC). *For $m \geqslant 4$, there is no resolute voting rule on SPOC profiles that is Condorcet-consistent and satisfies participation.*

The proof is a straightforward adaptation of the proofs of Moulin [1988] and Brandt et al. [2017]: most of the profiles in those proofs are SPOC. Somewhat more care needs to be taken while lifting the impossibility for $m = 4$ to $m > 4$ while maintaining the SPOC property.

Proof of Theorem 5. Let $m \geqslant 4$ and write $A = \{a_1, \dots, a_{m-3}, b, c, d\}$. Consider the circle $C = a_1 \lhd \dots \lhd a_{m-3} \lhd b \lhd c \lhd d \lhd a_1$. In the following, for brevity, we write $\vec{a}bcd$ for the linear order $a_1 \succ \dots \succ a_{m-3} \succ b \succ c \succ d$. The following profiles are all SPOC on C:

$$
\begin{aligned}
P_0 &= \phantom{2 \cdot \vec{a}bdc + {}} 2 \cdot \vec{a}bcd + 3 \cdot bcd\vec{a} + 3 \cdot d\vec{a}bc + 2 \cdot cd\vec{a}b \\
P_1 &= 2 \cdot \vec{a}bdc + 2 \cdot \vec{a}bcd + 3 \cdot bcd\vec{a} + 3 \cdot d\vec{a}bc + 2 \cdot cd\vec{a}b \\
P_2 &= \phantom{2 \cdot \vec{a}bdc + {}} 2 \cdot \vec{a}bcd + 3 \cdot bcd\vec{a} + 3 \cdot d\vec{a}bc + 2 \cdot cd\vec{a}b + 2 \cdot cdb\vec{a} \\
P_3 &= 2 \cdot \vec{a}bdc + 2 \cdot \vec{a}bcd + 3 \cdot bcd\vec{a} \\
P_4 &= 2 \cdot \vec{a}bdc + 2 \cdot \vec{a}bcd \phantom{{} + 3 \cdot bcd\vec{a}} + 3 \cdot d\vec{a}bc + 2 \cdot cd\vec{a}b \\
P_5 &= \phantom{2 \cdot \vec{a}bdc + {}} 2 \cdot \vec{a}bcd + 3 \cdot bcd\vec{a} \phantom{{} + 3 \cdot d\vec{a}bc} + 2 \cdot cd\vec{a}b + 2 \cdot cdb\vec{a} \\
P_6 &= \phantom{2 \cdot \vec{a}bdc + 2 \cdot \vec{a}bcd + 3 \cdot bcd\vec{a} + {}} 3 \cdot d\vec{a}bc + 2 \cdot cd\vec{a}b + 2 \cdot cdb\vec{a}
\end{aligned}
$$

Let f be a voting rule defined on SPOC profiles satisfying Condorcet-consistency and participation. By Condorcet-consistency, we have $f(P_3) = a_1$, $f(P_4) = d$, $f(P_5) = b$, and $f(P_6) = c$.

Write $A_{\vec{a}} = \{a_1, \dots, a_{m-3}\}$. Consider P_0, and assume for now that $f(P_0) \in A_{\vec{a}} \cup \{b\}$. Then, since f satisfies participation, also $f(P_1) \in A_{\vec{a}} \cup \{b\}$. By Condorcet-consistency, $f(P_3) = a_1$. Since P_1 can be obtained from P_3 by adding $d\vec{a}bc$ and $cd\vec{a}b$ voters, participation implies that $f(P_1) \in \{c, d, a_1\}$. Hence $f(P_1) = a_1$. But, by Condorcet-consistency, $f(P_4) = d$; and since P_1 can be obtained from P_4 by adding $bcd\vec{a}$ voters, participation implies $f(P_1) \in \{b, c, d\}$, a contradiction.

Since we arrived at a contradiction, we must have $f(P_0) \notin A_{\vec{a}} \cup \{b\}$, and so $f(P_0) \in \{c, d\}$. Thus, by participation, also $f(P_2) \in \{c, d\}$. By Condorcet-consistency, $f(P_5) = b$, and thus by participation, $f(P_2) \in A_{\vec{a}} \cup \{d, b\}$, and thus $f(P_2) = d$. But by Condorcet-consistency, $f(P_6) = c$ and thus by participation, $f(P_2) \in A_{\vec{a}} \cup \{b, c\}$, a contradiction.

Since either case leads to a contradiction, there can be no voting rule f with the desired properties. □

As described in the next section, further impossibilities about tournament-based rules can be deduced from Lemma 6.

6. Kemeny's and Young's Rules

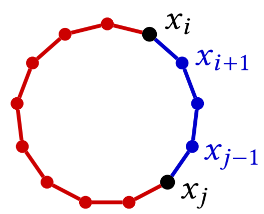
In this section, we will consider the problem of determining an election winner according to two well-known voting rules, Young's rule and Kemeny's rule, that are NP-hard to evaluate in general [Bartholdi III et al., 1989, Rothe et al., 2003, Hemaspaandra et al., 2005]. We will be interested to see whether these problems can be solved in polynomial time for SPOC profiles. We leave the complexity of Dodgson's rule for SPOC profiles for future work.

Kemeny's rule is a *rank aggregation* rule: Given a profile P over A, its aim is to produce a *consensus* ranking over A. Suppose r is a linear order over A. Its *Kemeny score* is $\sum_{i\in N} |v_i \cap r|$, the number of pairwise agreements of r with P. A *Kemeny ranking* is a linear order r with maximum Kemeny score. While it is NP-hard to find a Kemeny ranking [Bartholdi III et al., 1989], this problem is easy for single-peaked profiles whose transitive majority relation is easily seen to give rise to a Kemeny ranking. For SPOC preferences, the situation is less clear: the Condorcet paradox profile $(x \succ_1 y \succ_1 z, y \succ_2 z \succ_2 x, z \succ_3 x \succ_3 y)$ on 3 alternatives is SPOC, so SPOC does not guarantee a transitive majority relation. In fact, SPOC does not guarantee *anything at all* about the majority relation.

If P is a profile of linear orders, and $x, y \in A$, we write $n_{x,y} = |i \in N : x \succ_i y|$ for the number of voters who prefer x to y. The *majority margin* of x over y is $m_{x,y} = n_{x,y} - n_{y,x}$. Note that $m_{x,y} > 0$ if and only if a strict majority of voters prefer x to y. Note that $m_{x,y} \in \mathbb{Z}$ has the same parity as $|N|$ (since $m_{x,y} = n_{x,y} - (n - n_{x,y}) = 2n_{x,y} - n$), and so either all majority margins are even, or all are odd. The collection $(m_{x,y})_{x,y\in A}$ is known as a *weighted majority tournament.* McGarvey's [1953] theorem (and its refinement by Debord [1987]) states that any such collection of integers, all of the same parity, is induced as the weighted majority tournament of some profile P of linear orders. We can show that the same result holds when we additionally require that P is SPOC.

Lemma 6 (McGarvey's theorem for SPOC)**.** *All (weighted) majority tournaments are inducible by SPOC profiles.*

Proof. Fix a circle C with $x_1 \lhd x_2 \lhd \dots \lhd x_m$. For any $x_i, x_j \in A$ consider the profile P_{x_i,x_j} consisting of the following two votes which are single-peaked on C, with subscripts taken modulo m:



$$x_{i+1} \succ \dots \succ x_{j-1} \succ x_i \succ x_j \succ x_{j+1} \succ \dots \succ x_{i-1}$$
$$x_{i-1} \succ \dots \succ x_{j+1} \succ x_i \succ x_j \succ x_{j-1} \succ \dots \succ x_{i+1}$$

The profile P_{x_i,x_j} induces a majority tournament with all margins 0 except that $m_{x_i,x_j} = -m_{x_j,x_i} = 2$.

Suppose we are given a collection $(m_{x,y})_{x,y\in A}$ of even integers. Then consider the profile P which contains, for each pair x_i, x_j with $m_{x_i,x_j} > 0$, exactly $m_{x_i,x_j}/2$ copies of the profile P_{x_i,x_j}. Then P induces $(m_{x,y})_{x,y\in A}$, and P is SPOC on C.

Suppose we are given a collection $(m_{x,y})_{x,y\in A}$ of odd integers. Then consider the profile P containing one voter with $x_1 \succ \dots \succ x_m$, and also for each pair x_i, x_j with $i < j$, if $m_{x_i,x_j} > 0$ then $(m_{x_i,x_j} - 1)/2$ copies of P_{x_i,x_j}, and if $m_{x_i,x_j} < 0$ then $(m_{x_i,x_j} + 1)/2$ copies of P_{x_j,x_i}. Then P induces $(m_{x,y})_{x,y\in A}$, and P is SPOC on C. □

It is well-known that Kemeny scores only depend on the weighted majority tournament of a profile. Since the profiles in the proof of McGarvey's theorem above can be produced in polynomial time, the hardness of Kemeny in the general case carries over.

Theorem 7. *Finding a Kemeny ranking is NP-hard, even for SPOC preferences.*

Indeed, by the same argument essentially all negative (axiomatic or computational) results about voting rules based on (weighted) tournaments (see Brandt et al., 2016, Fischer et al., 2016) still hold restricted to SPOC preferences.

Young's rule. Given a profile P over A, an alternative $c \in A$ is a *Condorcet winner* if for every $b \in A \setminus \{c\}$, a majority of voters in P strictly prefers c to b. The *Young score* of an alternative $c \in A$ is the minimum number of voters that have to be deleted from P so that c becomes a Condorcet winner. Then, Young's rule selects all alternatives with minimum Young score as winners. It is known that Young winners can be found in polynomial time for single-peaked preferences [Brandt et al., 2015], since in this case Condorcet winners always exists when the number of voters n is odd; and the case with n even is also handled easily.

Because SPOC does not guarantee the existence of a Condorcet winner, a different approach is needed. We will use the interpretation of SPOC in terms of intervals of the underlying circle to give a polynomial-time algorithm that calculates the Young score of every alternative; clearly this algorithm can then be run repeatedly to find a Young winner. Of course, our algorithm also works for preferences single-peaked on a line; while the algorithm of Brandt et al. [2015] returns only a Young winner, our algorithm can find the Young *score* of any given alternative.[2] Note that precise definitions of Young scores differ slightly: sometimes it is only required that an alternative be made a *weak* Condorcet winner through voter deletion; our algorithm can be easily adapted for this alternative definition.

Theorem 8. *For SPOC profiles, the Young score of an alternative can be computed in $O(mn^2)$ time.*

Proof. We fix an axis $\lhd \in C$ that starts with the alternative a whose Young score we want to compute; let $a \lhd b \lhd \cdots \lhd c$ (b is the candidate right of a, c is the rightmost candidate). We partition voters into two sets: $N_1 = \{i \in N : b \succ_i a\}$ and $N_2 = N \setminus N_1$. Since P is SPOC, for any voter i, the set $I_i := \{d \in A : d \succ_i a\}$ forms an interval of $\lhd$. Voters in N_1 correspond to intervals containing b; voters in N_2 correspond to intervals containing c but not b, and to empty intervals. Figure 2 illustrates the situation: For each voter i, an arc indicates the set I_i. The red arcs on the right belong to voters from N_1, and the blue arcs on the left belong to voters from N_2.

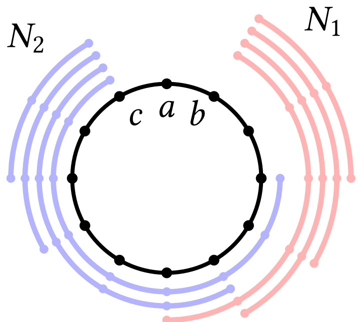


Figure 2: Illlustration of the proof of Theorem 8, for a profile with eight voter. For each voter, an arc indicates the set of alternatives preferred to a.

The idea behind our algorithm is that if there are voters i and j with $I_i \subseteq I_j$, then it is at least as profitable (for purposes of making a the Condorcet winner) to remove voter j as to remove voter i. Now note that the intervals I_i of voters in N_1 are nested by set inclusion, and similarly for voters

[2] Fitzsimmons and Hemaspaandra [2019, Theorem 10] give an algorithm for calculating Dodgson scores in single-peaked profiles.

in N_2. Thus, we let N_1^{-r} and N_2^{-s} denote the subsets of N_1 and N_2 obtained by deleting, respectively, the r and s voters from N_1 and N_2 that have the r and s largest (with respect to set inclusion) intervals I_i. Because of the nesting property, *if* there is a way of deleting r and s voters from N_1 and N_2 that makes a the Condorcet winner, then the deletions giving N_1^{-r} and N_2^{-s} also make a the Condorcet winner.

These observations suggest the following simple algorithm: For every pair (r, s) with $0 \leqslant r \leqslant |N_1|$ and $0 \leqslant s \leqslant |N_2|$, we check whether a is the Condorcet winner in $N_1^{-r} \cup N_2^{-s}$. We return a pair (r^*, s^*) with $r^* + s^*$ minimum for which this is the case. Then the Young score of a is $r^* + s^*$. If no such pair exists, the Young score of a is infinite.

To see that this algorithm can be run in $O(mn^2)$ time, we show how to check in $O(m)$ time whether a is the Condorcet winner in $N_1^{-r} \cup N_2^{-s}$. To do so, we precompute for every $x \in A \setminus \{c\}$, $0 \leqslant r \leqslant |N_1|$, and $0 \leqslant s \leqslant |N_2|$ the numbers

$$d_r^1(x) = |\{i \in N_1^{-r} : a \succ_i x\}| - |\{i \in N_1^{-r} : x \succ_i a\}|,$$
$$d_s^2(x) = |\{i \in N_2^{-s} : a \succ_i x\}| - |\{i \in N_2^{-s} : x \succ_i a\}|.$$

Note that a is a Condorcet winner in $N_1^{-r} \cup N_2^{-s}$ if and only if for all $x \in A \setminus \{c\}$ it holds that $d_r^1(x) + d_s^2(x) > 0$. The quantities $d_r^1(x)$ and $d_s^2(x)$ can be precomputed in $O(mn^2)$ time. Verifying whether $d_r^1(x) + d_s^2(x) > 0$ requires constant time and hence $O(m)$ time for every $x \in A \setminus \{c\}$. □

7. Multi-Winner Elections

Much recent work has studied voting rules that select not a single winner, but a *committee* $W \subseteq A$ of candidates, where $|W| = k$ has some desired size k (see, e.g., a recent survey by Faliszewski et al. [2017]). Depending on the context, we may wish this committee to have different properties. For example, we may aim for a representative committee in which as many voters as possible have a good representative, or we may aim for a proportional committee in which subgroups of the voters are represented by committee members in proportion to the subgroup size. Many of the commonly-studied multi-winner rules optimise an objective function over the set of all committees. Unsurprisingly, many of them are NP-hard to evaluate. In this section, we show that several popular rules can be evaluated in polynomial time when preferences are single-peaked on a circle.

Chamberlin and Courant [1983] introduced a rule that aims for a committee that represents as many voters as well as possible. It is usually defined for profiles of linear orders. According to this rule, each voter i is *represented* by i's favourite (highest-ranked) alternative in W; suppose this is $c_i \in W$. Then, we take the 'utility' of voter i to be the Borda score (i.e., position counting from the bottom) of c_i in i's ranking. The Chamberlin–Courant rule selects a committee of size k that maximises the sum of voter utilities. By replacing Borda scores by other scoring vectors, we obtain a whole family of rules. The class of OWA-based rules, as defined below, is a further generalization of this idea.

Finding a winning committee under the Chamberlin–Courant rule is known to be NP-hard for Borda scores [Lu and Boutilier, 2011]. Betzler et al. [2013] showed that this problem becomes easy when the input profile is single-peaked. Their algorithm starts by running a recognition algorithm for single-peakedness on the input to obtain an underlying axis $\lhd$ on which the profile is single-peaked. Then, they run a dynamic programming algorithm which constructs an optimal committee. Roughly, this dynamic program successively considers left prefixes of the axis $\lhd$, and constructs an optimal committee using only candidates from the prefix. Unfortunately, it is unclear how to extend this approach to preferences single-peaked on a circle, since a circle does not have a left endpoint where we could start the dynamic program.

Thus, we follow a different approach: We design an integer linear programming (ILP) formulation encoding the winner determination problem. We then show that the matrix of coefficients appearing in the constraints of this ILP is *totally unimodular* whenever the input profile is SPOC. A well-known result states that ILPs with totally unimodular constraint matrices are optimally solved by their LP relaxations [Hoffman and Kruskal, 1956], and can thus be solved in polynomial time.

This approach works not only for the Chamberlin–Courant rule, but for a large class of rules introduced by Skowron et al. [2016], called *OWA-based rules* (OWA stands for *ordered weighted average*). Let us describe this class of rules. We will give a definition that works for weak order inputs, and so this class includes rules that work for linear order profiles, and for approval profiles. Given a preference profile, as a first step the rule converts preferences into numerical scores, using a positional scoring system. Let us describe formally how this is done. Suppose that $\succcurlyeq$ is a weak order over A. Then we can uniquely partition $A = A_1 \cup \cdots \cup A_q$ into disjoint non-empty sets such that $A_1 \succ \cdots \succ A_q$ and such that $a \sim b$ for all $a, b \in A_r$ for $r \in [q]$. The sets A_r are called the *indifference classes* of the weak order $\succcurlyeq$. Now, for an alternative $a \in A$, if $a \in A_r$, the *rank* of a in $\succcurlyeq$ is r, and we write $\text{rank}_{\succcurlyeq}(a) = r$. Thus, the alternatives with rank 1 are the most-preferred alternatives. If we are given a profile P, then we write $\text{rank}_i(a)$ for $i \in N$ and $a \in A$ for the rank of a in voter i's preferences. A *score vector* is a vector $\mathbf{s} \in \mathbb{R}^m$ such that $s_1 \geqslant s_2 \geqslant \cdots \geqslant s_m$. Common examples are $\mathbf{s} = (m-1, m-2, \dots, 0)$ for Borda scores and $\mathbf{s} = (1, 0, \dots, 0)$ for plurality scores. Given such a score vector $\mathbf{s}$, we say that voter $i \in N$ assigns the score $s_{\text{rank}_i(a)}$ to alternative $a \in A$, and we write $\mathbf{s}(i, a) = s_{\text{rank}_i(a)}$. This is the standard definition when preferences are given by linear orders. If a voter submits an approval ballot, and we use plurality scores, then the voter assigns score 1 to all approved alternatives and score 0 to the remaining alternatives. Note that whenever $a \succcurlyeq_i b$ then $\mathbf{s}(i, a) \geqslant \mathbf{s}(i, b)$.

The utility a voter derives from a committee under an OWA-based rule will be a linear combination of the scores assigned to the candidates in the committee, and these values are calculated using an OWA operator. A weight vector $\boldsymbol{\alpha} \in \mathbb{R}^k$ defines an *ordered weighted average* (OWA) operator as follows: Given any vector $\mathbf{x} \in \mathbb{R}^k$, first sort the entries of $\mathbf{x}$ into non-increasing order, so that $x_{\sigma(1)} \geqslant \ldots \geqslant x_{\sigma(k)}$; second, apply the weights: the ordered weighted average of $\mathbf{x}$ with weights $\boldsymbol{\alpha}$ is given by $\boldsymbol{\alpha}(\mathbf{x}) := \sum_{i=1}^k \alpha_i x_{\sigma(i)}$. For example, if $\boldsymbol{\alpha} = (1, 0, \dots, 0)$, then $\boldsymbol{\alpha}(\mathbf{x}) = x_{\sigma(1)} = \max_{i \in [k]} x_i$, so that this operator returns the maximum of the vector $\mathbf{x}$. Alternatively, if $\boldsymbol{\alpha} = (1, 1, \dots, 1)$, then $\boldsymbol{\alpha}(\mathbf{x}) = \sum_{i=1}^k x_{\sigma(i)} = \sum_{i=1}^k x_i$, so that this operator gives the sum of the numbers in $\mathbf{x}$.

Given a profile P, a scoring vector $\mathbf{s}$, and an OWA operator $\boldsymbol{\alpha}$, we define the utility of a committee $W = \{c_1, \dots, c_k\}$ as

$$U(\mathbf{s}, \boldsymbol{\alpha}, W) = \sum_{i \in N} \boldsymbol{\alpha}(\mathbf{s}(i, c_1), \dots, \mathbf{s}(i, c_k)).$$

Then the OWA-based multi-winner rule based on $\mathbf{s}$ and $\boldsymbol{\alpha}$ outputs a committee W of size k for which $U(\mathbf{s}, \boldsymbol{\alpha}, W)$ is maximum.

For example, choosing $\boldsymbol{\alpha} = (1, 0, \dots, 0)$ and $\mathbf{s} = (m-1, m-2, \dots, 0)$ (Borda scores) gives us the Chamberlin–Courant rule, where each voter derives as utility the score of their favourite committee member. Choosing $\boldsymbol{\alpha} = (1, 1, 0, \dots, 0)$ gives us an analogue of Chamberlin–Courant where voters obtain as utility the sum of the scores of their favourite *two* members of the committee (this rule is sometimes known as 2-Borda). An OWA-based rule applied to profiles with approval votes with $\boldsymbol{\alpha} = (1, \frac{1}{2}, \dots, \frac{1}{k})$ and plurality scores $\mathbf{s} = (1, 0, 0, \dots)$ gives us Proportional Approval Voting (PAV). Thus, OWA-based rules generalise both Chamberlin–Courant and PAV.

Our polynomial-time result will only work for *non-increasing* OWA vectors $\boldsymbol{\alpha}$ with $\alpha_1 \geqslant \dots \geqslant \alpha_k$. For example, this excludes the rule where voters are represented by their *least*-favourite committee member, or by their median committee member. While such rules may be sensible in some contexts [Skowron, 2015], this restriction seems mild for most contexts.

Next, let us give an overview about total unimodularity. A matrix $A = (a_{ij})_{ij} \in \mathbb{Z}^{m \times n}$ with $a_{ij} \in \{-1, 0, 1\}$ is called *totally unimodular* if every square submatrix B of A has $\det B \in \{-1, 0, 1\}$. (The rows and columns of B need not occur contiguously in A.) The following results are well-known. Proofs and much more about their theory can be found in the textbook by Schrijver [1998].

Theorem 9 (see Schrijver, 1998, Theorem 19.1)**.** *Suppose $A \in \mathbb{Z}^{m \times n}$ is a totally unimodular matrix, $b \in \mathbb{Z}^m$ is an integral vector of right-hand sides, and $c \in \mathbb{Q}^n$ is an objective vector. Then the linear program*

$$\max c^T x \text{ subject to } Ax \leqslant b \tag{P}$$

has an integral optimum solution, which is a vertex of the polyhedron $\{x : Ax \leqslant b\}$. Thus, the integer linear program

$$\max c^T x \text{ subject to } Ax \leqslant b, x \in \mathbb{Z}^n \tag{IP}$$

is solved optimally by its linear programming relaxation (P).

An optimum solution to (IP) can be found in polynomial time [Maurras et al., 1981, Tardos, 1986]. We will now state some elementary results about totally unimodular matrices, which allows us to build new matrices from old.

Proposition 10 (see Schrijver, 1998, chapter 19)**.** *If A is totally unimodular, then so is*

1. *its transpose A^T,*
2. *the matrix $[A \mid -A]$ obtained from A by appending the negated columns of A,*
3. *the matrix $[A \mid I]$ where I is the identity matrix,*
4. *any matrix obtained from A through permuting or deleting rows or columns.*

In particular, from (3) and (4) it follows that appending a unit column $(0, \dots, 1, \dots, 0)^T$ will not destroy total unimodularity. Further, using these transformations, we can see that Theorem 9 remains true even if we add to (P) constraints giving lower and upper bounds to some variables, if we replace some of the inequality constraints by equality constraints, or change the direction of an inequality.

Recall that a binary matrix $A = (a_{ij}) \in \{0, 1\}^{m \times n}$ has the *consecutive ones property* if its columns can be permuted such that the 1-entries of each row form an interval. The key result that will allow us to connect single-peaked preferences to total unimodularity is as follows:

Proposition 11 (see Schrijver, 1998, page 279)**.** *Every binary matrix with the consecutive ones property is totally unimodular.*

We remark that by a celebrated result of Seymour [1980], it is possible to decide in polynomial time whether a given matrix is totally unimodular, though we do not use this fact.

We are now ready to prove our main result, that OWA-based rules are easy to compute for SPOC profiles.

Theorem 12. *Given a SPOC profile P, and an OWA-based rule specified by a scoring vector* $\mathbf{s}$ *and a non-increasing OWA operator* $\boldsymbol{\alpha}$*, a winning committee can be found in polynomial time.*

Proof. We begin by showing the result for single-peaked profiles, and later show how to modify the argument for SPOC profiles.

Let P be a preference profile, and let k be the target committee size. Consider the following integer linear program, whose optimal solutions correspond to winning committees under the OWA-based rule with operator $\boldsymbol{\alpha}$ and scoring vector $\mathbf{s}$. In the program, for each $r = 2, \dots, m$, we write $s'_r = s_r - s_{r-1}$, and we write $s'_1 = s_1$. Thus, for every $r \in [m]$, we have that $s_r = \sum_{p=1}^{r} s'_p$.

$$\text{maximise} \quad \sum_{i \in N} \sum_{\ell \in [k]} \sum_{r \in [m]} \alpha_\ell \cdot s'_r \cdot x_{i,\ell,r} \qquad \text{(OWA-IP)}$$

$$\text{subject to} \quad \sum_{c \in A} y_c = k \qquad (2)$$

$$\sum_{\ell \in [k]} x_{i,\ell,r} \leqslant \sum_{c\,:\,\mathrm{rank}_i(c) \leqslant r} y_c \qquad \text{for } i \in N,\ r \in [m] \qquad (3)$$

$$x_{i,\ell,r} \in \{0,1\} \qquad \text{for } i \in N,\ \ell \in [k],\ r \in [m]$$

$$y_c \in \{0,1\} \qquad \text{for } c \in A$$

Every feasible solution $((x_{i,\ell,r})_{i,\ell,r}, (y_c)_c)$ to the ILP corresponds to a committee $W = \{c \in A : y_c = 1\}$. Due to the constraint $\sum_{c \in A} y_c = k$, we have that $|W| = k$, so this is a committee of the required size. Next suppose that $S = ((x_{i,\ell,r})_{i,\ell,r}, (y_c)_c)$ is an optimal solution to the ILP. We may assume that under S, all constraints (3) are satisfied with equality, since otherwise we could set additional variables $x_{i,\ell,r}$ to 1 without affecting feasibility, and without lowering the objective value of the solution (because the coefficient $\alpha_\ell \cdot s'_r$ of $x_{i,\ell,r}$ is non-negative). Further, this operation does not change the committee W. Now fix a voter $i \in N$ and a rank $r \in [m]$. Suppose that there are L candidates in W which voter i places in rank r or better, i.e.,

$$L = |W \cap \{c \in A : \mathrm{rank}_i(c) \leqslant r\}|.$$

By our assumption that the constraint (3) is satisfied in S with equality, exactly L of the variables $x_{i,\ell,r}$ for $\ell \in [k]$ are set to 1 in S. By our assumption that the vector $\boldsymbol{\alpha}$ is non-increasing, the coefficients $\alpha_\ell \cdot s'_r$ of $x_{i,\ell,r}$ in the objective function are non-increasing as ℓ goes from 1 to k. Hence, we may assume without loss of generality that in S, we have

$$x_{i,1,r} = \dots = x_{i,L,r} = 1 \quad \text{and} \quad x_{i,L+1,r} = \cdots x_{i,k,r} = 0.$$

Then it follows that for $i \in N$, $\ell \in [k]$, $r \in [m]$, we have

$$x_{i,\ell,r} = 1 \quad \text{if and only if} \quad W \text{ contains at least } \ell \text{ candidates } c \text{ with } \mathrm{rank}_i(c) \leqslant r.$$

Fix $i \in N$ and $\ell \in [k]$. Write $W = \{c_1, \dots, c_k\}$ so that $c_1 \succcurlyeq_i \dots \succcurlyeq_i c_k$. Then it follows that, for every $r \in [m]$,

$$x_{i,\ell,r} = 1 \quad \text{if and only if} \quad \text{rank}_i(c_\ell) \leqslant r.$$

(If $x_{i,\ell,r} = 1$, then W contains at least ℓ candidates c with $\text{rank}_i(c) \leqslant r$, and so in particular $c_1, \dots, c_\ell$ must have rank r or better. Conversely, if $\text{rank}_i(c_\ell) \leqslant r$ then $c_1, \dots, c_\ell$ all have rank r or better, so there are at least ℓ candidates in W with rank r or better, and so $x_{i,\ell,r} = 1$.) Hence, the utility of voter i in committee W is

$$\begin{aligned}\boldsymbol{\alpha}(s_{\text{rank}_i(c_1)}, \dots, s_{\text{rank}_i(c_k)}) &= \sum_{\ell \in [k]} \alpha_\ell \cdot s_{\text{rank}_i(c_\ell)} \\ &= \sum_{\ell \in [k]} \alpha_\ell \cdot \left(\textstyle\sum_{r=1}^{\text{rank}_i(c_\ell)} s'_r \right) \\ &= \sum_{\ell \in [k]} \sum_{r=1}^{\text{rank}_i(c_\ell)} \alpha_\ell \cdot s'_r \\ &= \sum_{\ell \in [k]} \sum_{r \in [m]} \alpha_\ell \cdot s'_r \cdot x_{i,\ell,r}.\end{aligned}$$

Summing over all $i \in N$, we see that the objective value of solution S to the ILP equals $U(\mathbf{s}, \boldsymbol{\alpha}, W)$, the total utility of the committee W under the OWA-based rule. Thus, the ILP correctly encodes the winner determination problem.

Next suppose that the profile P is single-peaked. Consider the matrix M of coefficients in the constraints of the ILP. Let M' be the submatrix consisting only of the columns corresponding to the variables $(y_c)_{c \in A}$. Then M' has one row consisting of only 1s (corresponding to constraint (2)), and for each $i \in N$ and $r \in [m]$ a row whose 1-entries encode the set $\{c \in A : \text{rank}_i(c) \leqslant r\}$. Note that each of these sets is a top-initial segment of the preference order of i, and hence (see Section 2) an interval of the axis on which P is single-peaked. Therefore M' is a consecutive ones matrix (with the columns ordered according to the axis). Thus M' is totally unimodular by Proposition 11. Now, the matrix M is obtained from M' by appending columns corresponding to the variables $x_{i,\ell,r}$. Each of these variables occurs in only 1 constraint of type (3) with coefficient ± 1 (the sign depends on how we rearrange constraint (3) to bring all variables to one side). Thus, the column of the variable $x_{i,\ell,r}$ is a (negative) unit column, and so by Proposition 10, the matrix remains totally unimodular after appending it. Thus, M is totally unimodular. (Technically, we also need to include the constraints $0 \leqslant x_{i,\ell,r} \leqslant 1$ and $0 \leqslant y_c \leqslant 1$, but these are unit rows which can again be added without destroying total unimodularity.) Thus, by Theorem 9, the ILP can be solved in polynomial time.

The above argument for total unimodularity does not go through if P is SPOC but not single-peaked, because then the matrix M' only has the *circular* ones property. However, we can rearrange the ILP in such a way that we can show total unimodularity. This is a standard technique described in a useful survey by Dom [2009, Sec 4.1.4]. Before we begin, let us note the following general fact: suppose we are given a system of constraints

$$f(\mathbf{x}) = 0 \text{ and } g_j(\mathbf{x}) \leqslant 0 \text{ for } j = 1, \dots, J.$$

If in this system we replace one or more of the constraints $g_j(\mathbf{x}) \leqslant 0$ by $g_j(\mathbf{x}) - f(\mathbf{x}) \leqslant 0$, then the set of feasible solutions $\mathbf{x}$ to the system does not change.

Let P be a SPOC preference profile. Using the algorithms from Section 3, find a circle C such that P is single-peaked on C, and take some $\lhd \in C$ arbitrarily. For $i \in N$ and $r \in [m]$, write $T_{i,r} = \{c \in A : \text{rank}_i(c) \leqslant r\}$. Then $T_{i,r}$ is a top-initial segment of i's vote. Since P is single-peaked on C, $T_{i,r}$ is an interval of C. Thus, either $T_{i,r}$ or $A \setminus T_{i,r}$ is an interval of $\lhd$. Define the sets

$$\Gamma_1 = \{(i,r) : i \in N, r \in [m] \text{ such that } T_{i,r} \text{ is an interval of } \lhd\},$$
$$\Gamma_2 = \{(i,r) : i \in N, r \in [m] \text{ such that } A \setminus T_{i,r} \text{ is an interval of } \lhd\} \setminus \Gamma_1.$$

Then Γ_1 and Γ_2 form a partition of $N \times [m]$. Now consider the following integer linear program:

$$\begin{aligned}
\text{maximise} \quad & \sum_{i \in N} \sum_{\ell \in [k]} \sum_{r \in [m]} \alpha_\ell \cdot s'_r \cdot x_{i,\ell,r} && \text{(OWA-IP-SPOC)} \\
\text{subject to} \quad & \sum_{c \in A} y_c - k \;=\; 0 && \text{(2')} \\
& \sum_{\ell \in [k]} x_{i,\ell,r} \;\leqslant\; \sum_{c\,:\,\text{rank}_i(c) \leqslant r} y_c \qquad \text{for } i \in N,\ r \in [m] \text{ with } (i,r) \in \Gamma_1 && \text{(3')} \\
& \sum_{\ell \in [k]} x_{i,\ell,r} \;\leqslant\; -\sum_{c\,:\,\text{rank}_i(c) > r} y_c + k \qquad \text{for } i \in N,\ r \in [m] \text{ with } (i,r) \in \Gamma_2 && \text{(3'')} \\
& x_{i,\ell,r} \;\in\; \{0,1\} \qquad \text{for } i \in N,\ \ell \in [k],\ r \in [m] \\
& y_c \;\in\; \{0,1\} \qquad \text{for } c \in A
\end{aligned}$$

The program (OWA-IP-SPOC) is very similar to the original program (OWA-IP). Note that constraint (2') is the same as (2) after rearranging. The constraints (3') are a selection of the constraints (3). Finally, constraints (3") are obtained from constraints (3) after subtracting constraint (2'). Since (2') is an equality constraint, by the earlier mentioned general fact, we see that (OWA-IP-SPOC) and (OWA-IP) have the same set of feasible solutions. They also have the same objective function, and therefore (OWA-IP-SPOC) also correctly encodes the problem of finding a winning committee.

Finally, we can prove that (OWA-IP-SPOC) is totally unimodular, establishing the result that a winning committee can be found in polynomial time for SPOC profiles. Again take the constraint matrix M and consider the submatrix M' corresponding to the variables $(y_c)_{c \in A}$. If we rearrange the columns of M' according to $\lhd$, then each row of M' consists of either of an interval of $+1$s surrounded by 0s, or of an interval of -1s surrounded by 0s (the latter arising from constraints (3")). Combining Propositions 11 and 10, we see that M' is totally unimodular. As before, M is obtained from M' by adding columns with a single non-zero entry, so M is also totally unimodular. □

We obtain immediately the following two corollaries:

Corollary 13. *For SPOC profiles, Chamberlin–Courant can be computed in polynomial time.*

Corollary 14. *For SPOC profiles, PAV can be computed in polynomial time.*

Corollaries 13 and 14 clearly also apply to single-peaked profiles. For Chamberlin–Courant, a corresponding results has already been established using dynamic programming [Betzler et al., 2013], but to the best of our knowledge this is the first polynomial-time algorithm for PAV on single-peaked profiles[3].

An interesting question is whether the method of Theorem 12 can be further generalized. Fluschnik et al. [2019] study a rule that is an analogue of PAV where utilities need not be binary. They show that this generalization of PAV remains hard to compute for single-peaked utilities, thereby establishing a limit on the generalizability of the method of Theorem 12. Another possible generalization would be towards participatory budgeting, by introducing costs for candidates and replacing the committee size k by a budget limit. Does winner determination of OWA-based rules remain easy on single-peaked or SPOC profiles in this setting? Since totally unimodular matrices can only include coefficients from $\{-1, 0, 1\}$, it seems unlikely that the packing constraint for non-unit costs can be implemented in this approach. However, we are not aware of a hardness result for this problem.

The ILP formulation (OWA-ILP) from the proof of Theorem 12 is of independent interest for computing OWA-based rules; for example, it seems to have proven useful in the empirical study of Faliszewski et al. [2018]. Indeed, the "algorithm" that we propose for computing OWA-based rules (i.e., solving the program (OWA-ILP) using an ILP solver) is correct for general preferences, and comes with a polynomial-time guarantee in case the algorithm's input is single-peaked. This is in contrast to other winner determination algorithms that exploit preference structure: most such algorithms are specialized, and do not work at all if their input fails to be appropriately structured.

8. Discussion and Open Problems

Our results show that restricted preference domains that behave unfavorably in terms of axiomatic properties might still be very useful for algorithmic purposes. Indeed, our algorithms for Young's rule and OWA-based committee selection rules demonstrate that it is possible to move to a larger class than single-peaked preferences while maintaining polynomial-time runtime bounds. Thus, our findings can be seen as a challenge to established algorithmic results based on restricted preferences: to which degree can their application domain be extended without resorting to super-polynomial algorithms? One open problem of this type asks whether Dodgson's rule can be evaluated in polynomial time for SPOC profiles.

Our definition of SPOC is not the only sensible definition. One alternative definition that would fit into the generalised notion of single-peakedness introduced by Nehring and Puppe [2007] is based on *shortest paths*: it requires that for every voter i and for every alternative x, there is a *shortest* path between $\text{top}(i)$ and x along which i's preferences are decreasing. (Without the word "shortest" this is equivalent to SPOC.) The impact of this alternative definition is that every voter's least-preferred alternative needs to be *antipodal* to the voter's peak; this is strictly more restrictive than SPOC. It would be interesting to see whether this smaller domain allows for a wider range of positive results than SPOC.

The SPOC domain is strictly larger the single-peaked domain. To quantify the actual gain in generality—or the increase in likelihood—it would be of interest to know the number of SPOC profiles for a given number of voters and candidates. This line of work has been pursued by Durand [2003], Lackner and Lackner [2017], and Chen and Finnendahl [2018] for single-peaked profiles and thus

[3]As PAV is based on dichotomous preferences, single-peakedness corresponds to the *candidate interval* restriction [Faliszewski et al., 2011, Elkind and Lackner, 2015].

could be compared with results for SPOC.

Another direction for future work would be to extend the SPOC concept to two (and more) dimensions – preferences single-peaked on a *sphere* – but this may be difficult since little is known even about extensions of single-peakedness on a *line* to two or more dimensions (see Sui et al. 2013). Further generalizations can be realized by considering *almost* SPOC profiles, mirroring the literature on almost single-peaked profiles [Elkind et al., 2012, Cornaz et al., 2012, 2013, Faliszewski et al., 2014, Bredereck et al., 2016, Erdélyi et al., 2017].

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A. Recognition Algorithm

In this section, we describe the algorithm of Fitzsimmons and Lackner [2019] for deciding whether a profile consisting of weak orders, at least one of which is a linear order, is single-peaked. This algorithm is the basis of Theorem 2 (showing that SPOC profiles of linear orders can be recognised in linear time) and is also the basis of the proof of Theorem 3 (characterisation by forbidden subprofiles), as we will see in Appendix B.

Algorithm 1 (also called the *guided algorithm*) takes as input a profile P of weak orders, at least one of which (the *guiding vote* v_g) is a linear order. The algorithm then attempts to construct an axis on which P is single-peaked. The axis is constructed from the outside in. The process is guided by v_g: the algorithm scans the alternatives in the ranking v_g from bottom to top, and decides at each step whether to place the alternative to the left or to the right.

Let us introduce some notation for understanding Algorithm 1. The algorithm labels alternatives so that $c_m \succ_g c_{m-1} \succ_g \cdots \succ_g c_1$. We write $C_{>i} = \{c_{i+1}, \ldots, c_m\}$. The algorithm represents the axis under construction as a list of alternatives. For two disjoint lists A_L, A_R of alternatives, $\langle A_L, A_R \rangle$ is the concatenation of the lists. Similarly, we write $\langle c_i, A_R \rangle$ and $\langle A_L, c_i \rangle$ for preprending and appending an alternative to a list. For a collection A' of alternatives, and a vote v_k, we write $\max_k(A')$ for an arbitrary alternative such that $\max_k(A') \succcurlyeq a$ for all $a \in A'$, and we write $\min_k(A')$ for an arbitrary alternative such that $a \succcurlyeq \min_k(A')$ for all $a \in A'$. When deciding how to place alternatives, the algorithm checks whether some of the conditions (R1), (R2), (L1), or (L2) apply to a specific voter v_k. These conditions are defined as follows:

$$c_i \succ_k \min_k(C_{>i}) \quad \text{and} \quad \max_k(A_L) \succ_k \min_k(C_{>i}) \tag{R1}$$

$$\max_k(C_{>i}) \succ_k c_i \quad \text{and} \quad \max_k(A_R) \succ_k c_i \tag{R2}$$

$$c_i \succ_k \min_k(C_{>i}) \quad \text{and} \quad \max_k(A_R) \succ_k \min_k(C_{>i}) \tag{L1}$$

$$\max_k(C_{>i}) \succ_k c_i \quad \text{and} \quad \max_k(A_L) \succ_k c_i \tag{L2}$$

Algorithm 1 Recognising SP

Input: a profile $P = (v_1, v_2, \dots, v_n)$ of weak orders, where v_g is a linear order
Output: Is P single-peaked?

Label alternatives such that $c_m \succ_g c_{m-1} \succ_g \cdots \succ_g c_1$
Set $A_L \leftarrow \langle\rangle$ and $A_R \leftarrow \langle c_1 \rangle$
for $i = 2, \dots, m$ **do**
  if for no voter $v_k \neq v_g$ we have either (R1) or (R2) **then**
    place c_i on the right: $A_R \leftarrow \langle c_i, A_R \rangle$
  else if for no voter $v_s \neq v_g$ we have either (L1) or (L2) **then**
    place c_i on the left: $A_L \leftarrow \langle A_L, c_i \rangle$
  else
    (for some v_k we have (R1) or (R2) and for some v_s we have (L1) or (L2))
    return P is not single-peaked
  end if
end for
return P is single-peaked on the axis $\lhd = \langle A_L, A_R \rangle$

To prove that Algorithm 1 is correct, we need to prove that (i) when it returns an axis $\lhd$, then P is single-peaked on $\lhd$, and that (ii) when it returns that P is not single-peaked, then P is not single-peaked on any axis. This is shown by Fitzsimmons and Lackner [2019]. Below, for completeness, we prove part (i). Part (ii) is not formally required for our purposes, since the correctness of the SPOC recognition algorithm only relies on (i).

Proposition 15 (Fitzsimmons and Lackner, 2019)**.** *Let P be a profile of weak orders, containing at least one linear order. If Algorithm 1 returns an axis $\lhd$, then P is single-peaked on $\lhd$.*

Proof. It is easy to see that the guiding vote v_g is single-peaked on $\lhd$, since any top-initial segment $\{c_i, c_{i+1}, \dots, c_m\}$ of v_g is an interval of $\lhd$ by construction. Suppose some other vote $v_k \neq v_g$ is not single-peaked on $\lhd$. Then there must be alternatives $a, b, c \in A$ such that $a \lhd b \lhd c$ and $a \succ_k b$ and $c \succ_k b$ (a *valley*).

We may assume without loss of generality that $c \succ_g a$ (otherwise we can rename alternatives so that the axis $\lhd$ is reversed). We perform a case analysis on the preferences of the guiding vote v_g, i.e., on the order in which Algorithm 1 has placed the alternatives:

- $a \prec_g b \prec_g c$ (alternatives are placed a first then b then c): Consider the iteration when b was placed, i.e., when $c_i = b$. At this point, a has already been placed, and since $a \lhd b \lhd c$, we have $a \in A_L$. Note that condition (L2) is now satisfied for v_k since $a \succ_k b$ and $c \succ_k b$. Hence, b must have been placed on the right. So the algorithm returns an axis with $a \lhd c \lhd b$, a contradiction.

- $a \prec_g c \prec_g b$ (alternatives are placed a first then c then b): Consider the iteration when c was placed, i.e., when $c_i = c$. As before, we have $a \in A_L$. Note that (R1) is satisfied for v_k and so c is placed on the left. So the algorithm returns an axis with $a \lhd c \lhd b$, a contradiction.

- $b \prec_g a \prec_g c$ (alternatives are placed b first then a then c): If alternatives are placed in this order, the algorithm cannot obtain an axis with $a \lhd b \lhd c$, where b is between a and c, a contradiction.

Since each case leads to a contradiction, v_k must in fact be single-peaked on $\lhd$. ◻

B. Characterisation by Forbidden Subprofiles

Theorem 3. A profile P of linear orders on A is *not* SPOC if and only if one of the following three cases occurs.

1. There are distinct alternatives $a, b, c, d, e \in A$ and voters v_i and v_j in P such that

$$\{a, b\} \succ_i \{c\} \succ_i \{d, e\},$$
$$\{a, e\} \succ_j \{c\} \succ_j \{d, b\}.$$

2. There are distinct alternatives $a, b, c, d \in A$ and voters v_i, v_j, and v_k in P such that

$$\{a, b\} \succ_i \{c, d\},$$
$$\{a, c\} \succ_j \{b, d\},$$
$$\{a, d\} \succ_k \{b, c\}.$$

3. There are distinct alternatives $a, b, c, d \in A$ and voters v_i, v_j, and v_k in P such that

$$\{a, b\} \succ_i \{c, d\},$$
$$\{b, c\} \succ_j \{a, d\},$$
$$\{c, a\} \succ_k \{b, d\}.$$

Proof. Sufficiency was proven in Section 4. Here, we prove necessity. Suppose that P is a profile of linear orders which is not SPOC. Then the recognition algorithm of Theorem 2, run on input P, will return that P is not SPOC (this follows from Propositions 1 and 15). We prove that if the algorithm gives a negative answer, then P must contain one of the forbidden subprofiles.

The algorithm of Theorem 2 first constructs the profile $\bar{P}$ obtained from P by slicing at some alternative z, where z is an alternative which is ranked last by some voter (say v_1) in P. Then the upper part resulting from slicing v_1 is the same as v_1, and hence is a linear order. Call the voter in $\bar{P}$ corresponding to this upper part g. We label alternatives such that $c_m \succ_g c_{m-1} \succ_g \cdots \succ_g c_2 \succ_g z$.

After constructing the sliced profile $\bar{P}$, the algorithm of Theorem 2 invokes Algorithm 1 from Appendix A and runs it on $\bar{P}$ to determine whether $\bar{P}$ is single-peaked. From Proposition 1, we know that $\bar{P}$ is single-peaked if and only if P is SPOC. Since we assumed that P is not SPOC, $\bar{P}$ is not single-peaked, and hence Algorithm 1 will return that $\bar{P}$ is not single-peaked (by Proposition 15). We will analyse all possible scenarios under which Algorithm 1 can give a negative answer, and show that in each case one of the three conditions of Theorem 3 applies. These conditions all refer to the profile P rather than $\bar{P}$, and it will be convenient to transform them into statements about $\bar{P}$, which we do in two steps.

First, note that the conditions of Theorem 3 are closed under reversing some of the votes in the condition. We formalise this observation in Lemma 16 below. For a voter v_j, we write $x_1 \leftrightarrow_j x_2 \leftrightarrow_j \cdots \leftrightarrow_j x_k$ if either $x_1 \succ_j x_2 \succ_j \cdots \succ_j x_k$ or $x_k \succ_j \cdots \succ_j x_2 \succ_j x_1$. Where we do not care about the relative order of some alternatives, we write sets as before: $A_1 \leftrightarrow_j A_2 \leftrightarrow_j A_3$ means that either $A_1 \succ_j A_2 \succ_j A_3$ or $A_3 \succ_j A_2 \succ_j A_1$.

Lemma 16. *Condition 1 of Theorem 3 is satisfied if and only if there are distinct alternatives $a, b, c, d, e \in A$ and voters v_i and v_j in P such that*

$$\{a, b\} \leftrightarrow_i \{c\} \leftrightarrow_j \{d, e\},$$
$$\{a, e\} \leftrightarrow_j \{c\} \leftrightarrow_j \{d, b\}.$$

One of conditions 2 or 3 of Theorem 3 is satisfied if and only if there are distinct alternatives $a, b, c, d \in A$ and voters v_i, v_j, and v_k in P such that

$$\{a, b\} \leftrightarrow_i \{c, d\},$$
$$\{a, c\} \leftrightarrow_j \{b, d\},$$
$$\{a, d\} \leftrightarrow_k \{b, c\}.$$

Proof of Lemma 16. An easy case analysis. □

Second, we give conditions on the sliced profile $\bar{P}$ whose occurrence implies that one of the conditions of Lemma 16 holds. From now on, we will label the voters as $\bar{P} = (\bar{v}_1, \ldots, \bar{v}_{2n})$. The voter corresponding to the upper part of the guiding vote will also be called g. For a voter $\bar{v}_i$, we write $a \succ_i b$ if a is ranked higher than b in $\bar{v}_i$.

Lemma 17. *One of the conditions of Theorem 3 (equivalently, of Lemma 16) is satisfied if one of the following sufficient conditions is satisfied:*

1. *There are distinct alternatives $a, b, c, d \in A \setminus \{z\}$ and $e \in A$ and voters $\bar{v}_i$ and $\bar{v}_j$ in $\bar{P}$ such that*

$$\{a, b\} \succ_i \{c\} \succ_i \{e, d\},$$
$$\{a, d\} \succ_j \{c\} \succ_j \{e, b\}.$$

2. *There are distinct alternatives $a, b, c, d \in A \setminus \{z\}$ and voters $\bar{v}_i$, $\bar{v}_j$, and $\bar{v}_k$ in $\bar{P}$ such that*

$$\{a, b\} \succ_i \{c, d\}$$
$$\{a, c\} \succ_j \{b, d\}$$
$$\{a, d\} \succ_k \{b, c\}.$$

3. *There are distinct alternatives $a, b, c \in A \setminus \{z\}$ and $d \in A$ and voters $\bar{v}_i$, $\bar{v}_j$, and $\bar{v}_k$ in $\bar{P}$ such that*

$$\{a, b\} \succ_i \{c, d\},$$
$$\{b, c\} \succ_j \{a, d\},$$
$$\{c, a\} \succ_k \{b, d\}.$$

4. *There are distinct alternatives $a, b, c, d \in A \setminus \{z\}$ and voters $\bar{v}_i$ and $\bar{v}_j$ in $\bar{P}$ such that*

$$\{a, b\} \succ_i c \sim_i d,$$
$$\{a, c\} \succ_j b \sim_i d.$$

5. *There are distinct alternatives $a, b, c \in A \setminus \{z\}$ and voters $\bar{v}_i$, $\bar{v}_j$, and $\bar{v}_k$ in $\bar{P}$ such that*

$$\{a, b\} \succ_i \{c, z\},$$
$$\{a, c\} \succ_j \{b, z\},$$
$$a \succ_k z \sim_k b \sim_k c.$$

Proof of Lemma 17. We start with a useful observation: Suppose that $\bar{v}_i$ is a voter in $\bar{P}$ with $x_1 \succ_i \cdots \succ_i x_k$ for some $x_1, \ldots, x_k \in A \setminus \{z\}$. Consider the voter v_{i^*} in P from which $\bar{v}_i$ was obtained. Then, for v_{i^*}, we have $x_1 \leftrightarrow_{i^*} \cdots \leftrightarrow_{i^*} x_k \leftrightarrow_{i^*} z$, by the definition of slicing.

Suppose conditions 1, 2, or 3 of this lemma hold. Then apply the above observation to each of the voters, obtaining one of the cases of Lemma 16.

Suppose condition 4 holds. Then consider the voters v_{i^*} and v_{j^*} in P from which $\bar{v}_i$ and $\bar{v}_j$ were obtained. Because $\bar{v}_i$ is indifferent between c and d, in the vote v_{i^*}, the alternatives c and d are separated from a and b by the alternative z (by the definition of slicing). Thus, we have $\{a, b\} \leftrightarrow_{i^*} z \leftrightarrow_{i^*} \{c, d\}$. Similarly, we have $\{a, c\} \leftrightarrow_{j^*} z \leftrightarrow_{j^*} \{b, d\}$. Thus, by the first part of Lemma 16, we are done.

Suppose condition 5 hold. Consider the voters v_{i^*}, v_{j^*}, and v_{k^*} in P from which $\bar{v}_i$, $\bar{v}_j$, and $\bar{v}_k$ were obtained. Then we have $\{a, b\} \leftrightarrow_{i^*} \{c, z\}$ and $\{a, c\} \leftrightarrow_{j^*} \{b, z\}$. Further, in the vote v_{k^*}, the alternatives b and c are separated from a by the alternative z, and thus $a \leftrightarrow_{k^*} z \leftrightarrow_{k^*} \{b, z\}$. Thus, by the second part of Lemma 16, we are done. □

As we said above, because P is not SPOC, $\bar{P}$ is not single-peaked, and hence Algorithm 1 returns a negative answer when run on $\bar{P}$. Thus, line 10 of Algorithm 1 is executed. Inspecting the if-clauses, this means that there is some iteration i (in which we are placing alternative c_i on the axis) when

$$\text{for some } \bar{v}_k \text{ in } \bar{P} \text{ we have (R1) or (R2), and for some } \bar{v}_s \text{ in } \bar{P} \text{ we have (L1) or (L2).} \qquad (*)$$

By case analysis, we will prove that (∗) implies that one of the conditions of Lemma 17 holds. This suffices to prove the theorem. The cases we consider are:

A. condition R1 and L1 holding for the same voter ($\bar{v}_k = \bar{v}_s$),

B. condition R2 and L2 holding for the same voter ($\bar{v}_k = \bar{v}_s$),

C. condition R1 and L1 holding for different voters ($\bar{v}_k \neq \bar{v}_s$),

D. condition R2 and L2 holding for different voters ($\bar{v}_k \neq \bar{v}_s$),

E. condition R1 and L2 holding for the same voter ($\bar{v}_k = \bar{v}_s$), and

F. condition R1 and L2 holding for different voters ($\bar{v}_k \neq \bar{v}_s$).

Note that R1 and L1 are symmetric; the same holds for R2 and L2. Hence we can omit the analogues of cases E and F for condition R2 and L1.

For each of the cases A to F, we need to identify voters and alternatives to which one of the conditions of Lemma 17 apply. To do so, we will repeatedly apply the following rules, which can be proven by inspecting Algorithm 1. These rules are valid for every $\bar{v}_i$ in $\bar{P}$, and they hold at each iteration i of Algorithm 1, with A_L and A_R as defined by the algorithm at the start of the iteration.

- **(g)** — We have $\{c_{i+1}, \ldots, c_m\} \succ_g \{c_i\} \succ_g (A_L \cup A_R) \setminus \{z\} \succ_g \{z\}$, by the choice of vote g, its role in the guiding algorithm, and our labelling of alternatives.
- **(z)** — Let $a, b \in A$. If $a \succ_i b$, then $a \succ_i z$. Furthermore, if $a \sim_i b$, then $a \sim_i z$. This is because z is ranked last in all votes in $\bar{P}$ and $\bar{P}$ contains top orders.
- **(top)** — Let $a, b, c \in A$ with $a \neq b$. If $a \sim_i b$ and $c \succ_i a$, then $c \succ_i b$. This is because in all votes in $\bar{P}$, indifferences only occur among bottom-ranked alternatives (they are *top orders*).
- **(ind)** — Let $\ell \in A_L$ and $a \in \{c_i, \ldots, c_m\}$. If $\ell \succ_i a$, then $a \succcurlyeq_i r$ for all $r \in A_R$. Similarly if left and right are reversed. This follows from the inductive correctness proof of the guided algorithm; the algorithm would not have placed ℓ and r in such a way that ℓ, a, r form a valley.
- **(ind2)** — Let $\ell_1, \ell_2 \in A_L$ such that ℓ_1 is placed left of ℓ_2 in A_L, and let $a \in \{c_i, \ldots, c_m\}$. Then $\ell_2 \succcurlyeq_i \ell_1$ or $\ell_2 \succcurlyeq_i a$. Similarly if left and right are reversed. This follows from the inductive correctness proof of the guided algorithm; otherwise the algorithm would not have placed ℓ_2 on the left side as ℓ_1, ℓ_2, a would form a valley.

We will use pictures to display various relations between alternations. Arrows $a \longrightarrow b$ signify that $a \succ b$; labels on arrows indicate the rule used to deduce this relation. Similarly, a dashed line a- - -b indicates $a \sim b$.

We are now ready to go through our case analysis.

A. Condition R1 holds for voter $\bar{v}_k$, and condition L1 holds for voter $\bar{v}_k$.

   This case is impossible due to (ind): we have both $\max_k(A_L) \succ_k \min_k(C_{>i})$ and $\max_k(A_R) \succ_k \min_k(C_{>i})$.

B. Condition R2 holds for voter $\bar{v}_k$, and condition L2 holds for voter $\bar{v}_k$.

   This case is also impossible due to (ind): we have both $\max_k(A_L) \succ_k c_i$ and $\max_k(A_R) \succ_k c_i$.

C. Condition R1 holds for voter $\bar{v}_k$, and condition L1 holds for voter $\bar{v}_s$, with $\bar{v}_k \neq \bar{v}_s$.

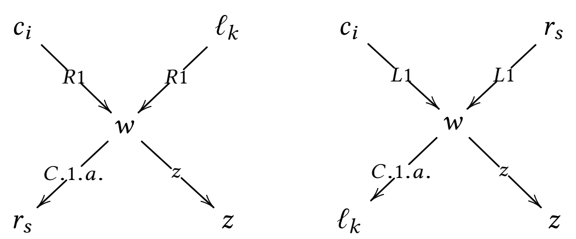
   Let $w_k = \min_k(C_{>i})$, let $w_s = \min_s(C_{>i})$, let $\ell_k = \max_k(A_L)$, and let $r_s = \max_s(A_R)$.

   By R1, $c_i \succ_k w_k$ and $\ell_k \succ_k w_k$. By L1, $c_i \succ_s w_s$ and $r_s \succ_s w_s$.

   C.1. Suppose $w_k = w_s =: w$. From (ind), we have $w \succcurlyeq_k r_s$ and $w \succcurlyeq_s \ell_k$.

   C.1.a. Suppose $w \succ_k r_s$ and $w \succ_s \ell_k$. Then the following relations can be deduced:

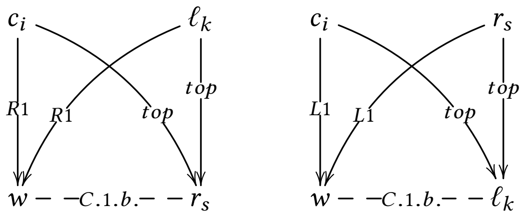
   sliced vote $\bar{v}_k$: sliced vote $\bar{v}_s$:



   This corresponds to case 1 in Lemma 17 and thus we encounter case 1 in the theorem statement.

C.1.b. Suppose $w \sim_k r_s$ and $w \sim_s \ell_k$. Then

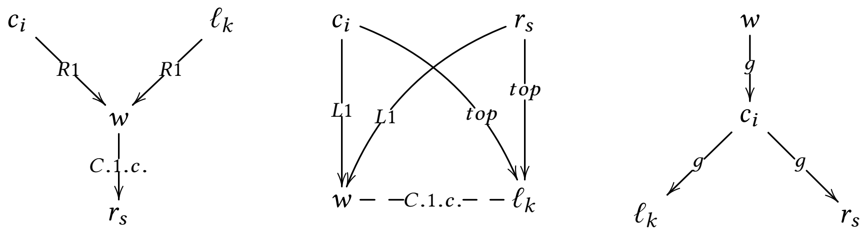
sliced vote $\bar{v}_k$: sliced vote $\bar{v}_s$:



This corresponds to case 4 in Lemma 17.

C.1.c. Suppose $w \succ_k r_s$ and $w \sim_s \ell_k$.

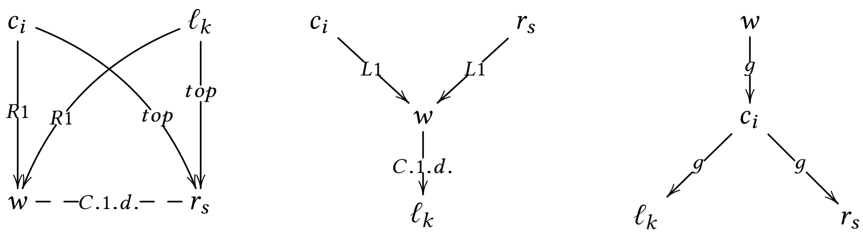
sliced vote $\bar{v}_k$: sliced vote $\bar{v}_s$: guiding vote $\bar{g}$:



This corresponds to case 2 in Lemma 17.

C.1.d. Suppose $w \sim_k r_s$ and $w \succ_s \ell_k$. Then, symmetrically to C.1.c.,

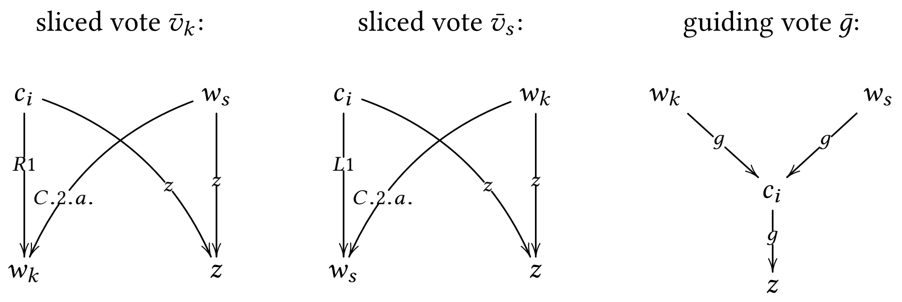
sliced vote $\bar{v}_k$: sliced vote $\bar{v}_s$: guiding vote $\bar{g}$:



This corresponds to case 2 in Lemma 17.

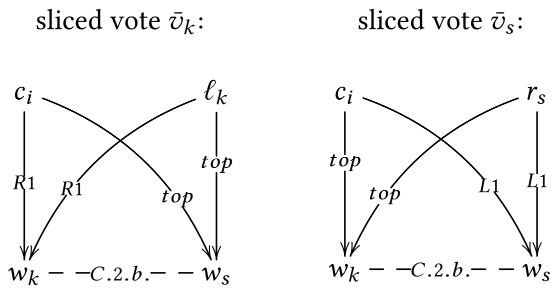
C.2. Suppose $w_k \neq w_s$. By definition of w_s and w_k, we have $w_s \succcurlyeq_k w_k$ and $w_k \succcurlyeq_s w_s$.

C.2.a. Suppose $w_s \succ_k w_k$ and $w_k \succ_s w_s$. Then we have



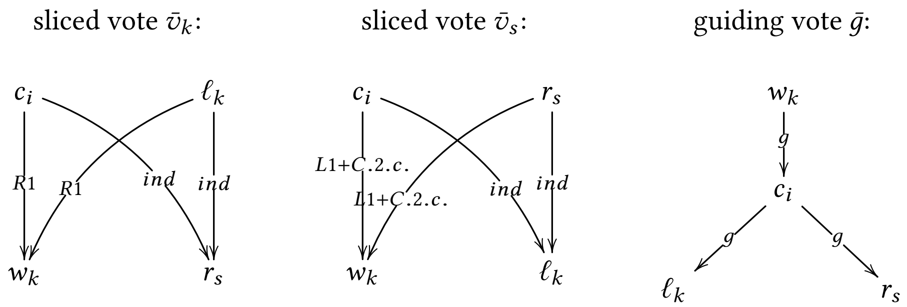
This corresponds to case 3 in Lemma 17.

C.2.b. Suppose $w_s \sim_k w_k$ and $w_k \sim_s w_s$. Then we have



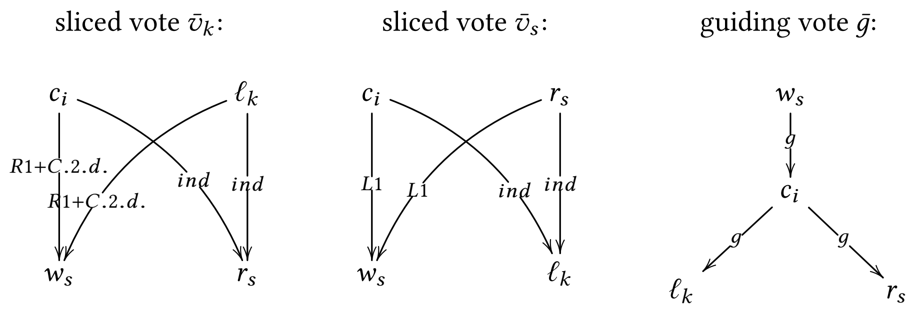
This corresponds to case 4 in Lemma 17.

C.2.c. Suppose $w_s \succ_k w_k$ and $w_k \sim_s w_s$. Since $\ell_k \succ_k w_k$ by R1, we have $w_k \succcurlyeq_k r_s$ by (ind), and thus $\ell_k \succ_k r_s$ by transitivity. Similarly we obtain the other arrows labelled (ind) below, each involving a use of transitivity.



This corresponds to case 2 in Lemma 17. The arrows labeled with "L1+C.2.c." are obtained by using L1 for $c_i \succ_s w_s$ and $r_s \succ_s w_s$ and the assumption of case C.2.c., $w_k \sim_s w_s$.

C.2.d. Suppose $w_s \sim_k w_k$ and $w_k \succ_s w_s$. Then, symmetric to 2.c.,



This corresponds to case 2 in Lemma 17.

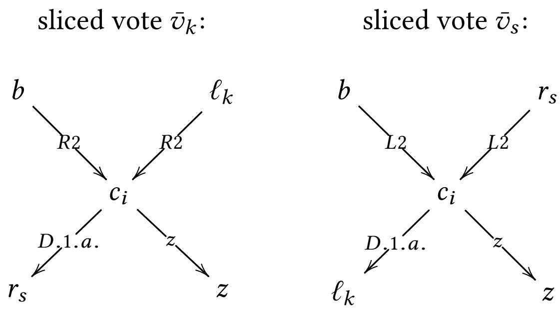
D. Condition R2 holds for voter $\bar{v}_k$, and condition L2 holds for voter $\bar{v}_s$, with $\bar{v}_k \neq \bar{v}_s$.

Let $b_k = \max_k(C_{>i})$, let $b_s = \max_s(C_{>i})$, let $\ell_k = \max_k(A_L)$, and let $r_s = \max_s(A_R)$.

By R2, $b_k \succ_k c_i$ and $\ell_k \succ_k c_i$. By L2, $b_s \succ_s c_i$ and $r_s \succ_s c_i$.

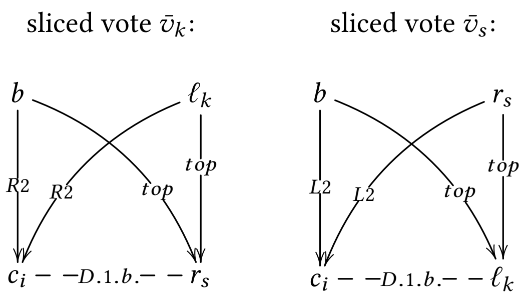
D.1. Suppose $b_k = b_s =: b$. The case D.1. is similar to case C.1.; we present it for completeness.

D.1.a. Suppose $c_i \succ_k r_s$ and $c_i \succ_s \ell_k$. Then:



This corresponds to case 1 in Lemma 17.

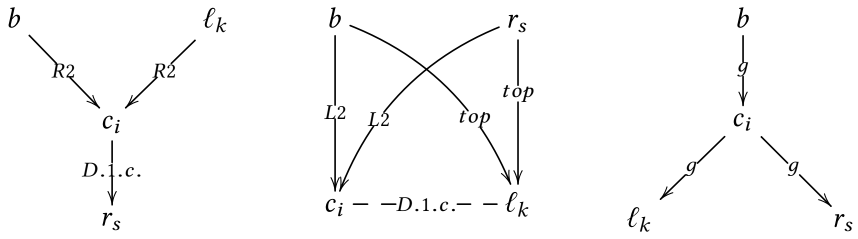
D.1.b. Suppose $c_i \sim_k r_s$ and $c_i \sim_s \ell_k$. Then



This corresponds to case 4 in Lemma 17.

D.1.c. Suppose $c_i \succ_k r_s$ and $c_i \sim_s \ell_k$.

sliced vote $\bar{v}_k$: sliced vote $\bar{v}_s$: guiding vote $\bar{g}$:



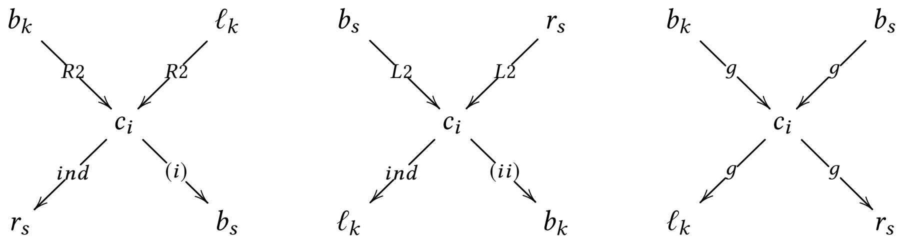
This corresponds to case 2 in Lemma 17.

D.1.d. Suppose $c_i \sim_k r_s$ and $c_i \succ_s \ell_k$. This case is symmetric to D.1.c.

D.2. Suppose $b_k \neq b_s$.

D.2.a. Suppose that either (i) $c_i \succ_k b_s$ or (ii) $c_i \succ_s b_k$. Then we can deduce one of the first two depicted relations ($\bar{v}_k$ or $\bar{v}_s$) as well as the guiding vote $\bar{g}$:

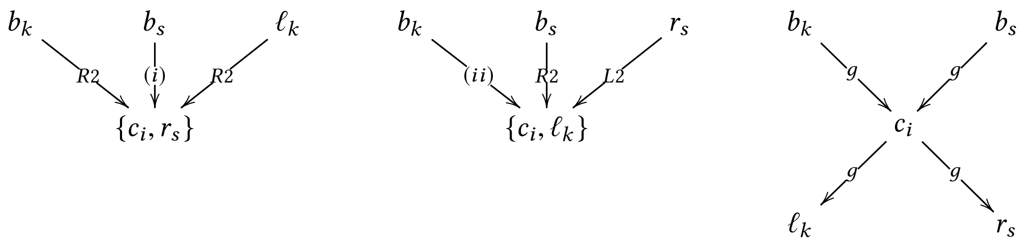
sliced vote $\bar{v}_k$: sliced vote $\bar{v}_s$: guiding vote $\bar{g}$:



If $c_i \succ_k b_s$, then k and g correspond to case 1 in Lemma 17; if $c_i \succ_s b_k$, then s and g correspond to case 1 in Lemma 17.

D.2.b. Now suppose that (i) $c_i \prec_k b_s$ or (ii) $c_i \prec_s b_k$. As $\ell_k \succ_k c_i$ (from R2), by (ind) we have $c_i \succeq_k r_s$. Analogously, since $r_s \succ_s c_i$ (from L2), by (ind) we have $c_i \succeq_s \ell_k$. Thus:

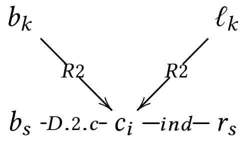
sliced vote $\bar{v}_k$: sliced vote $\bar{v}_s$: guiding vote $\bar{g}$:



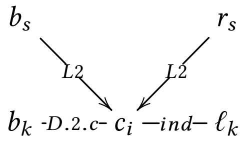
In case (i), this corresponds to case 2 in Lemma 17 when considering $\{b_s, c_i, \ell_k, r_s\}$. In case (ii), this corresponds to case 2 in Lemma 17 when considering $\{b_k, c_i, \ell_k, r_s\}$.

D.2.c. Otherwise, we have $c_i \sim_k b_s$ and $c_i \sim_s b_k$. By (ind), from $\ell_k \succ_k c_i$ (R2), and the fact that $\bar{v}_k$ is a top order, we have $c_i \sim_k b_s \sim_k r_s$. Analogously, by (ind), from $r_s \succ_s c_i$ (L2), and the fact that $\bar{v}_s$ is a top order, we have $c_i \sim_s b_k \sim_s \ell_k$.

sliced vote $\bar{v}_k$:



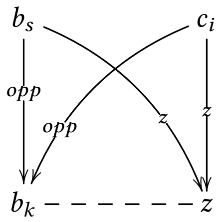
sliced vote $\bar{v}_s$:



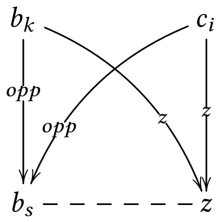
We see that $\bar{v}_k$ and $\bar{v}_s$ hold little information. Thus, we consider their opposite sliced part, i.e., if $\bar{v}_k$ is a sliced upper part, then we consider the lower part, or vice versa. Let $\bar{v}'_k$ and $\bar{v}'_s$ be the corresponding opposite parts.

Now suppose that a candidate $y \neq z$ is in the lowest level of $\bar{v}_k$. Then y is not in the lowest level of $\bar{v}'_k$. Also, if $x \succ_k y$, then x is in the lowest level of $\bar{v}'_k$, and hence $y \succ_{k'} x$. Call this observation (opp). From this we get:

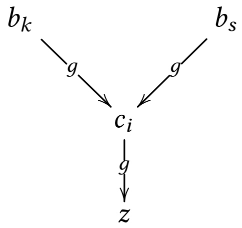
opposite sliced vote $\bar{v}'_k$:



opposite sliced vote $\bar{v}'_s$:



guiding vote $\bar{g}$:



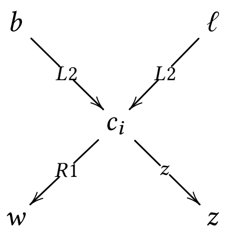
This corresponds to case 3 in Lemma 17.

E. Condition R1 holds for voter $\bar{v}_k$, and condition L2 holds for voter $\bar{v}_k$.

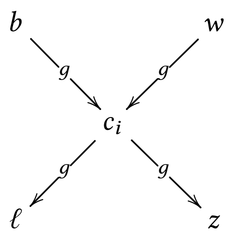
   Let $w = \min_k(C_{>i})$, let $b = \max_k(C_{>i})$, let $\ell = \max_k(A_L)$.

   By R1, $c_i \succ_k w$ and $\ell \succ_k w$. By L2, $b \succ_k c_i$ and $\ell \succ_k c_i$. Note that $b \succ_k c_i \succ_k w$ and so $b \neq w$. We have

   sliced vote $\bar{v}_k$:



   guiding vote $\bar{g}$:



   This corresponds to case 1 in Lemma 17.

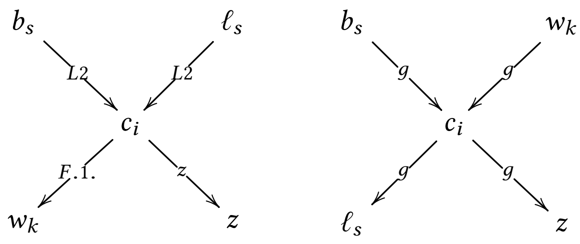
F. Condition R1 holds for voter $\bar{v}_k$, and condition L2 holds for voter $\bar{v}_s$, with $\bar{v}_k \neq \bar{v}_s$.

   Let $w_k = \min_k(C_{>i})$, let $b_s = \max_s(C_{>i})$, let $\ell_k = \max_k(A_L)$, and let $\ell_s = \max_s(A_L)$.

   By R1, $c_i \succ_k w_k$ and $\ell_k \succ_k w_k$. By L2, $b_s \succ_s c_i$ and $\ell_s \succ_s c_i$.

F.1. Suppose $c_i \succ_s w_k$. Since then $b_s \succ_s c_i \succ_s w_k$, we have $b_s \neq w_k$. Then we have

sliced vote $\bar{v}_s$: guiding vote $\bar{g}$:

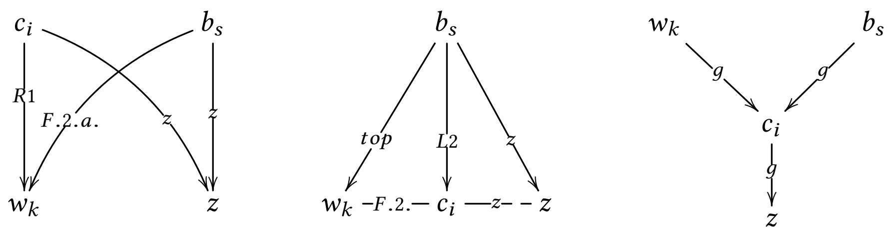


This corresponds to case 1 in Lemma 17.

F.2. Suppose $w_k \sim_s c_i$. Since $\bar{v}_s$ is a top order, and $b_s \succ_s c_i \sim_s w_k$, we must have $b_s \neq w_k$.

F.2.a. Suppose $b_s \succ_k w_k$. Then we have

sliced vote $\bar{v}_k$: sliced vote $\bar{v}_s$: guiding vote $\bar{g}$:



This corresponds to case 5 in Lemma 17.

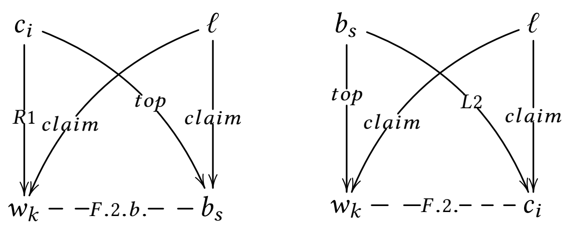
F.2.b. Suppose $w_k \sim_k b_s$.

Claim: There is $\ell \in \{\ell_s, \ell_k\}$ for which both $\bar{v}_s$ and $\bar{v}_k$ agree that ℓ is not in the bottom layer

If $\ell_s = \ell_k$, we are done by R1 and L2. So suppose $\ell_s \neq \ell_k$. Now, by choice of ℓ_s and ℓ_k, we have $\ell_s \succcurlyeq_s \ell_k$ and $\ell_k \succcurlyeq_k \ell_s$. But $\bar{v}_s$ and $\bar{v}_k$ are top orders, and $\ell_s \succ_s c_i$ and $\ell_k \succ_k w_k$, so actually $\ell_s \succ_s \ell_k$ and $\ell_k \succ_k \ell_s$. Next we show that it is not possible that both ℓ_k is in s's bottom layer and ℓ_s is in k's bottom layer. Suppose this was so. Suppose ℓ_s is placed by the algorithm to the left of ℓ_k. Then $\bar{v}_s$ has a valley: $\ell_s \succ_s \ell_k \prec_s b_s$. Otherwise, ℓ_k is placed by the algorithm to the left of ℓ_s. Then $\bar{v}_k$ has a valley: $\ell_k \succ_k \ell_s \prec_k c_i$. Both possibilities contradict (ind2).

With this choice of ℓ, we have

sliced vote $\bar{v}_k$: sliced vote $\bar{v}_s$:

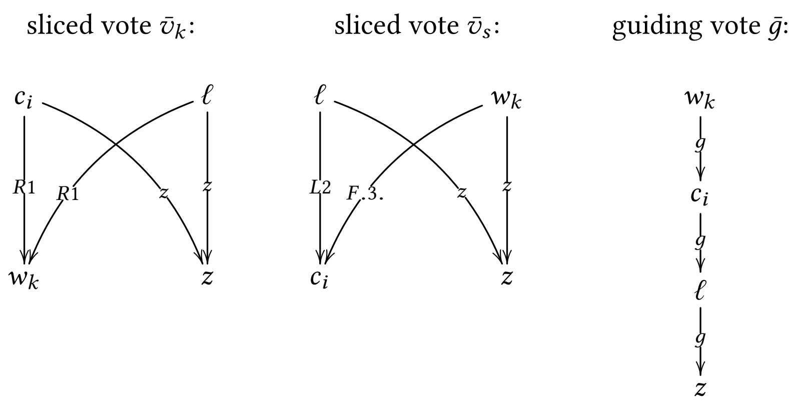


This corresponds to case 4 in Lemma 17.

F.2.c. Suppose $w_k \succ_k b_s$. This is impossible by choice of $w_k = \min_k(C_{>i})$.

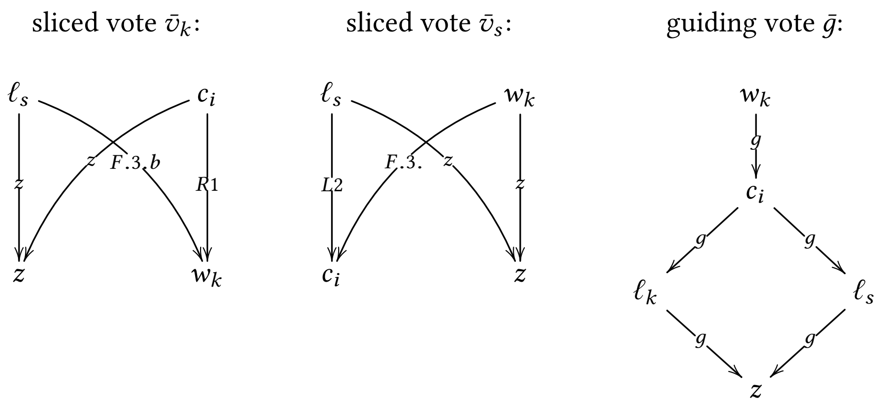
F.3. Suppose $w_k \succ_s c_i$. (Note we do not consider b_s in the following subcases, so it does not matter if $b_s = w_k$ or not.)

F.3.a. Suppose $\ell_s = \ell_k =: \ell$. Then we have



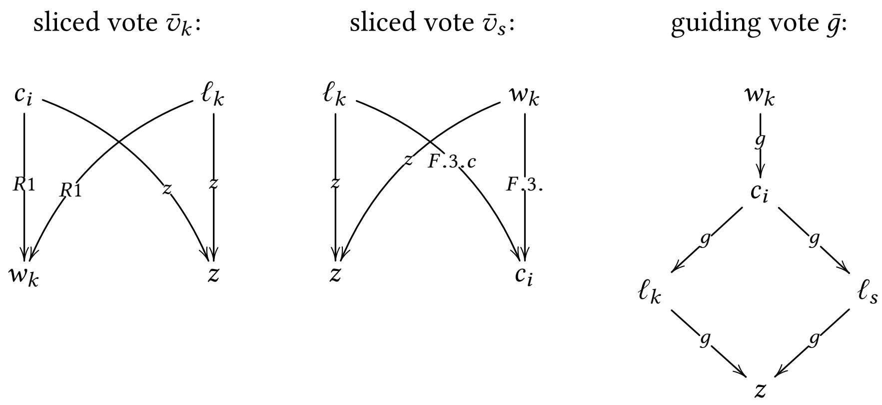
This corresponds to case 3 in Lemma 17.

F.3.b. Suppose ℓ_k lies to the left of ℓ_s (within A_L). By (ind2) it holds that $\ell_s \succcurlyeq_k \ell_k$ or $\ell_s \succcurlyeq_k c_i$. As $\ell_k \succ_k w_k$ and $c_i \succ_k w_k$ (both by R1), we can conclude that $\ell_s \succ_k w_k$. So we have



This corresponds to case 3 in Lemma 17.

F.3.c. Suppose ℓ_s lies to the left of ℓ_k (within A_L). By (ind2) it holds that $\ell_k \succeq_s w_k$ or $\ell_k \succeq_s \ell_s$. As $w_k \succ_s c_i$ (by F.3.) and $\ell_s \succ_s c_i$ (by L2), we can conclude that $\ell_k \succ_s c_i$. So we have



This corresponds to case 3 in Lemma 17.

We have shown that in all possible situations where the guided algorithm may return *no*, Lemma 17 is applicable and thus the theorem holds. □

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