Structural Tractability in Hedonic Games
Dominik Peters  Computer Science, University of Oxford, UK

Efficient Coalition Formation when Agents’ Preferences are Structured:

<table>
<thead>
<tr>
<th>FIND STABLE OUTCOMES</th>
<th>MAXIMISE SOCIAL WELFARE</th>
<th>ALLOCATE GOODS FAIRLY</th>
<th>FASTER ALGORITHMS FOR GAMES THAT MATTER</th>
</tr>
</thead>
</table>

### Hedonic Games: The Model

Finite set $N$ of agents, each $i \in N$ having preferences $\succ_i$ over groups of agents:

$$\{1, 2\} \succ_1 \{1, 2, 3\} \succ_1 \{1\} \succ_1 \{1, 3\}$$

**Outcome:** a partition $\pi$ of the agent set $N$.

### Solution Concepts

We want partition $\pi$ to be \textit{stable}. What could this mean?

- **Nash stable (NS):** no individual wants to change into another group.
- **Individually stable (IS):** no individual can change into a group in which all members welcome her.
- **Core stable (CR):** no group $S$ of agents all prefer $S$ to where they are in $\pi$.
- **Strict Core stable (SCR):** only require 1 deviator in $S$ to have a strict preference.

### Social Welfare: if agents have numerical \textit{utilities}, we can aim to find a partition that has high social welfare, i.e., the sum of the agents’ utilities is large.

### Envy-free: We can also define \textit{fairness} concepts: a partition is envy-free if no agent wishes to take the position of another agent.

### Computational Complexity: Bad News

**Given:** a hedonic game (concisely represented somehow)

**Question:** Does there exist a \textit{good} outcome in one of the senses above? If yes, find one.

- Large AI literature studies complexity of this question
- Almost always \textit{intractable}: questions are hard for NP or even for the second level of the polynomial hierarchy
- Previous attempts at identifying islands of tractability has focussed on restricting individual preferences, say by considering preferences obtained by adding, averaging, or minimising values.
- Peters and Elkind (IJCAI 2015): this approach can't work
- New approach needed: let’s try a \textit{structural} one

### Graphical Hedonic Games

**Idea:** Agents are part of a \textit{social network}, and we try to exploit the structure of the network topology.

Formally, we equip a hedonic game with an \textit{agent dependency graph} $G = (N, E)$ so that agents only care about whether they are together with their neighbours:

$$S \succ_i T \iff S \cap \text{Ngb}_G(i) \succ_i T \cap \text{Ngb}_G(i)$$

**Note:** Can always take the complete graph, but we usually want few edges.

### Structure in the Network Topology

- If the social network of a graphical hedonic game is structured in some way, algorithms may be able to use this structure.
- Throughout algorithmic graph theory, bounding \textit{treewidth} is a wildly successful technique for obtaining tractability.
- We report results for bounded treewidth and bounded max-degree below.
- Future work: Other structural restrictions: planar graphs? minor-free graphs? bipartite graphs?

### Results: Bounded Treewidth and Degree

**Theorem.** Deciding whether a graphical hedonic game admits a stable outcome is linear-FPT w.r.t. the treewidth $k$ and max-degree $d$ of the underlying graph; that is, the problem can be solved in time $O(f(k, d) \cdot n)$ where $f$ is a computable function.

**Proof method:** Define “HG-logic” to capture properties of hedonic games, translate to MSO, apply Courcelle’s theorem.

**Theorem.** Maximising utilitarian or egalitarian social welfare in a graphical hedonic game can be done in time $O(2^{kd^2} \cdot n)$. We can also find a Nash- or individually stable outcome in time $O(2^{kd^2} \cdot n)$.

**Theorem.** It is \textit{necessary} to bound the max-degree: finding a stable outcome remains NP-hard for graphical hedonic games of treewidth 2, but of unbounded degree.

### Application: Allocation of Indivisible Goods

- A collection of objects need to be allocated to agents
- Agents have preferences over bundles of objects.
- Goal: a fair allocation, or a welfare-maximising allocation (= combinatorial auctions).
- This model can be seen as a special case of hedonic games, where objects are agents that are indifferent between all outcomes.
- Positive results above transfer to this setting.

Presented at AAAI 2016, Phoenix, Arizona