Preferences Single-Peaked on Nice Trees

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Single-Peaked on a Tree

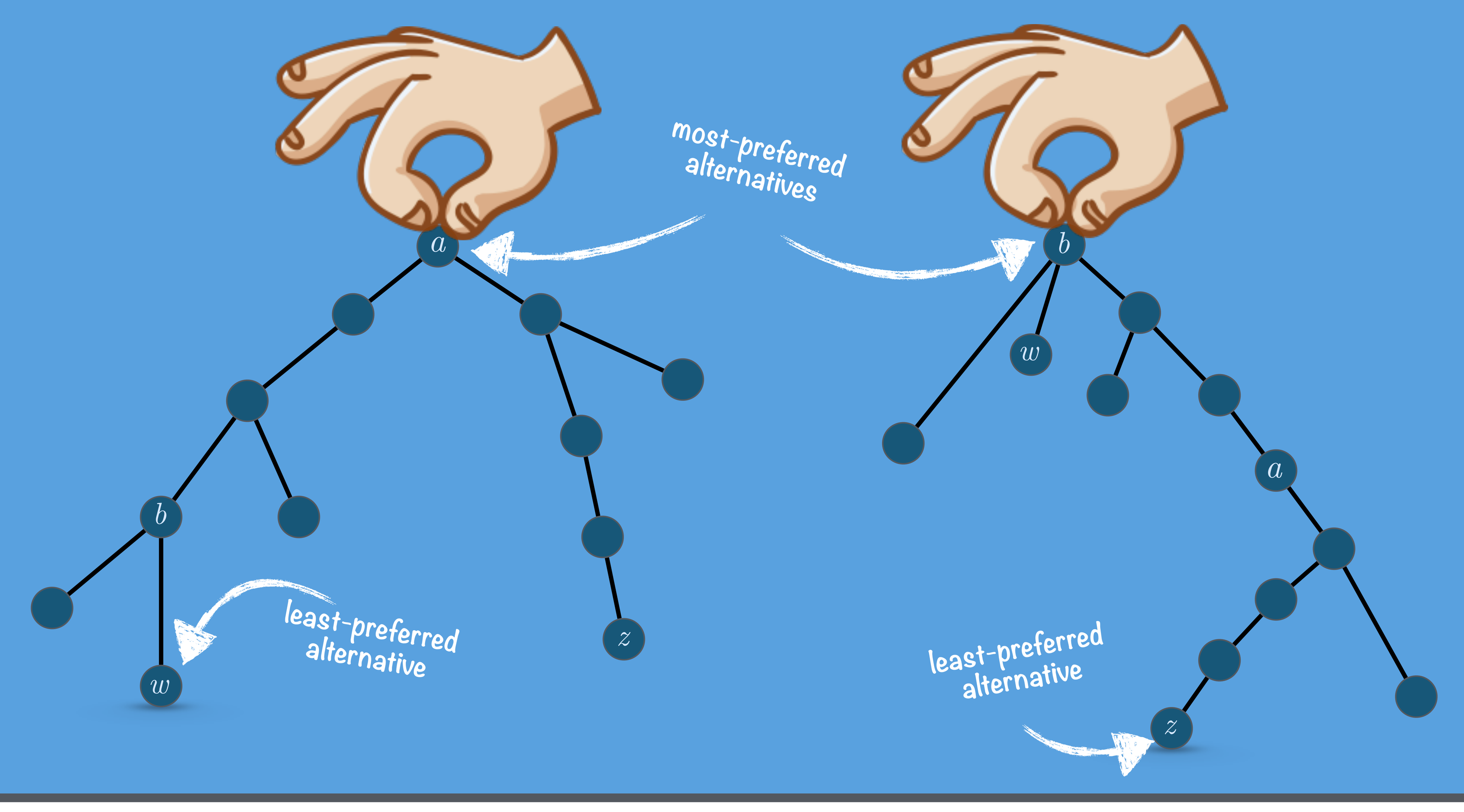
introduced by Demange (1982)

- more general than single-peaked on a line
- Condorcet winner guaranteed to exist
- allows faster algorithms for voting problems
- efficiently recognisable

Definitions. Let A be an m-element set of *alternatives*, and let V be an n-element set (or *profile*) of *votes*, i.e., strict preference orders over A. Let $T = (A,E)$ be a tree over A.

V is **single-peaked on T** iff
V is single-peaked on every path of T.

Equivalently, V is single-peaked on T if for every voter i and all numbers k, the set of i's k most preferred alternatives induces a (connected) subtree of T.



Trick's Recognition Algorithm

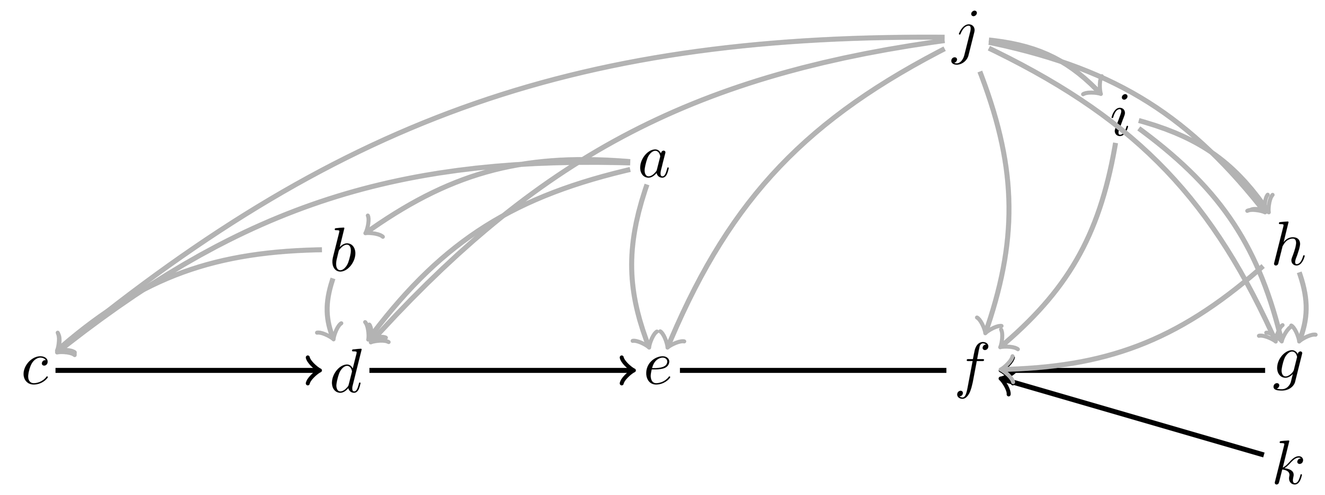
- Trick (1987) gives an $O(nm^2)$ algorithm for checking if a preference profile is single-peaked on a tree.
- If yes, it returns *some* tree that works.

Key insight behind algorithm: if an alternative appears in last position in some voter's preference, then the alternative must be a leaf (similar to standard single-peaked). Identify such a leaf alternative and *attach* it to an *arbitrarily* chosen other alternative, but without violating single-peakedness. Repeat.

- Some algorithms for voting problems that exploit the tree structure of preferences work better if the tree is "nice"
- For example, nice could mean having few leaves, or low degrees, or small pathwidth.

Finding *Nice* Trees (if possible)

- Question: *Can we modify Trick's algorithm to return a **nice** tree if that's possible?*
- Answer: *Yes, by keeping track of the decisions that the algorithm made.*
- In fact, we can concisely represent *all* trees on which a given profile is single-peaked using the "attachment digraph":



Instructions for use: select one outgoing arc for each alternative, forget orientations, and we obtain a tree on which the profile {kfedghcijba, dcbeafghijk, gfhiedcbajk} is single-peaked. All such trees can be obtained in this way. Note that the black arcs are forced, *and the grey arcs are* 'free'. *Crucially, among the grey arcs, the digraph is transitive.*

Recognition Algorithms

Theorem. Given a profile V that is single-peaked on a tree, we can find in polynomial time a suitable tree that among suitable tree has a minimum number of **leaves**, a minimum number of **internal vertices**, minimum **diameter**, minimum **max-degree**, minimum **path-width.**

Further, we can decide in polynomial time whether a given profile is single-peaked on a **line**, **star**, **caterpillar**, **lobster**, or on a **subdivision of a star**.

Theorem. However, it is NP-complete to decide, given a profile V and an unlabelled tree T with $|A|$ vertices, whether V is single-peaked on **some labelling** of T.
Also NP-hard to decide whether single-peaked on a **regular** tree.

Application: Committee Selection

- The Chamberlin-Courant rule is a popular voting rule for selecting an "optimal" committee of size k from the m candidates in A.
- Optimal committee minimises a *misrepresentation function.*
- The rule is NP-hard to evaluate in general.
- Yu, Chan, and Elkind (2013) show this problem becomes easier for preferences single-peaked on a tree with *few leaves.*
- This paper: also easy for trees with *few internal vertices.*

Theorem. Given a profile that is single-peaked on a tree with r internal vertices, we can find a winning committee of size k under the Chamberlin-Courant rule with Borda misrepresentation function in time poly$(n, m, (k+1)^r)$.

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