Preferences Single-Peaked on Nice Trees

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Single-Peaked on a Tree

introduced by Demange (1982)

more general than single-peaked on a line
Condorcet winner guaranteed to exist
allows faster algorithms for voting problems
efficiently recognisable

Definitions. Let $A$ be an $n$-element set of alternatives, and let $V$ be an $n$-element set (or profile) of votes, i.e., strict preference orders over $A$. Let $T = (A, E)$ be a tree over $A$.

$V$ is single-peaked on $T$ if
$V$ is single-peaked on every path of $T$.

Equivalently, $V$ is single-peaked on $T$ if for every voter $i$ and all numbers $k$, the set of $i$'s $k$ most preferred alternatives induces a (connected) subtree of $T$.

Finding Nice Trees (if possible)

- Question: Can we modify Trick’s algorithm to return a nice tree if that’s possible?
- Answer: Yes, by keeping track of the decisions that the algorithm made.
- In fact, we can concisely represent all trees on which a given profile is single-peaked using the “attachment digraph”:

Trick’s Recognition Algorithm

- Trick (1987) gives an $O(nm^2)$ algorithm for checking if a preference profile is single-peaked on a tree.
- If yes, it returns some tree that works.

Key insight behind algorithm: if an alternative appears in last position in some voter’s preference, then the alternative must be a leaf (similar to standard single-peaked). Identify such a leaf alternative and attach it to an arbitrarily chosen other alternative, but without violating single-peakedness. Repeat.

- Some algorithms for voting problems that exploit the tree structure of preferences work better if the tree is “nice”
- For example, nice could mean having few leaves, or low degrees, or small pathwidth.

Application: Committee Selection

- The Chamberlin-Courant rule is a popular voting rule for selecting an “optimal” committee of size $k$ from the $m$ candidates in $A$.
- Optimal committee minimises a misrepresentation function.
- The rule is NP-hard to evaluate in general.
- Yu, Chan, and Elkind (2013) show this problem becomes easier for preferences single-peaked on a tree with few leaves.
- This paper: also easy for trees with few internal vertices.

Theorem. Given a profile $V$ that is single-peaked on a tree, we can find in polynomial time a suitable tree that among suitable trees has a minimum number of leaves, a minimum number of internal vertices, minimum diameter, minimum max-degree, minimum path-width.

Further, we can decide in polynomial time whether a given profile is single-peaked on a line, star, caterpillar, lobster, or on a subdivision of a star.

Theorem. However, it is NP-complete to decide, given a profile $V$ and an unlabelled tree $T$ with $|A|$ vertices, whether $V$ is single-peaked on some labelling of $T$. Also NP-hard to decide whether single-peaked on a regular tree.